OPTIMIZATION OF BEAMS FOR LATERAL BUCKLING

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THESIS

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FOR LATERAL BUCKLING

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ABSTRACT

The treatise seeks to find the optimum shape of a thin beam under the influence of lateral buckling. The specific formulation is by means of the isoperimetric problem of the calculus of variations. An energy method approach yields the governing equations for the problem. An analytic solution was not obtained due to the nonlinearity and coupling conditions of the equations.

A variable height lateral buckling problem is formulated as an alternate attempt to find the optimum design. Through Bessel equation analysis numerous designs are obtained and the resulting buckling loads are calculated. The largest buckling load corresponds to a beam design which appears to be very close to the optimum shape.

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LIST OF SYMBOLS

A	cross sectional area
a	constant of integration
b	variable width
b_o	constant width
C_1	material constant
C_2	material constant
D	domain
E	Young's modulus
F	functional
f	function
G	shear modulus
h	variable height
h_o	constant height
I	integral function
i	vector subscript
J	polar moment of inertia
j	vector subscript
K	dimensionless constant
k	dependent variable subscript
L, ℓ	length
M	moment
N	column load
n	exponent for design
P	concentrated load
p_o	uniform load

s	surface
T	total potential energy
U	strain energy
V	potential energy external forces
V_o	volume
u,v,w	displacements
x,y,z	coordinate axes
α	constant of integration
β	Bessel equation parameter
γ	$(1 - 2x/\ell)$
δ	variation
ε	strain
ξ,η,ζ	principle axes
λ	Lagrangian multiplier
ν	Bessel function order
ρ	Bessel equation argument
σ	stress
ψ	specific strain energy

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I. INTRODUCTION

A. THE OPTIMIZATION PROBLEM

The performance of an elastic body is measured by the maximum load which the body is able to withstand. The optimization problem considered in this treatise is that of determining the specific design of an elastic structure such that the structure will exhibit a maximum performance capacity. Herein, the term design refers to the actual physical distribution of material within an elastic body.

The optimization problem is formulated by the calculus of variations. This method offers certain advantages in the development of a theory for optimization of an elastic body. The specific formulation of the problem is by means of the isoperimetric problem of the calculus of variations. By this method the problem becomes an extremization of the total potential energy of the system with a superimposed volume constraint. The extremization of the functional, augmented by Lagrangian multipliers, leads to a set of Euler equations which govern the design of the structure.

The calculus of variations is no new technique for obtaining an optimum design. Lagrange [1] formulated an optimum column problem by this means in 1773. He sought to determine the shape of a column for which the buckling load was largest among all columns of given length and volume. Although he failed in obtaining the correct solution, his

recognition of the problem belonging to the calculus of variations inspired others to continue research in this field. Clausen [2] later provided a solution to this problem for columns of circular cross sections. Keller [3] recently showed that the strongest column is not one of a circular cross section, but instead, is one which has an equilateral triangle as a cross section.

The isoperimetric problem for optimum design received an additional stimulus from Keller and Tadjbakbsh [4]. This work provided a variational technique for minimum volume, elastic design of columns. Taylor [5] treated the same problem by means of an energy method approach. In his work Taylor showed that the same governing equations may be obtained by direct extremization of a functional, which consists of the system potential energy and a volume constraint. Taylor's approach is the essential method used for optimization in this thesis. The method is particularly useful since it provides a simple and direct means for isoperimetric problem formulation.

An extension of the Taylor approach was accomplished by Salinas [6] and provides some conditions for the validity of the potential energy formulation. Additionally, the author demonstrates the equivalence of maximum load, minimum volume optimum designs and presents a general formulation for optimization of elastic structures.

B. SCOPE OF THE THESIS

Using the Taylor approach, the optimization problem is discussed and developed by use of Lagrangian multipliers and the calculus of variations. The resulting set of Euler equations are seen to be the system equilibrium equations and an optimality condition. Example problems are included to illustrate the usefulness of the Taylor method. These include a strength problem for a uniformly loaded beam and an optimum design problem for a particular case of column buckling.

The method is then extended to the case of a thin beam subjected to lateral buckling. Under lateral buckling, the beam is considered to buckle in a plane perpendicular to the vertical plane of loading. The buckling configuration consists of rotation and sidewise bending away from the horizontal axis of symmetry. The resulting equations governing an optimum design are found to be non-linear. In addition, the equations are coupled in rotation and bending. All attempts for an analytic solution proved futile.

An attempt to find the optimum shape has been made by assuming geometrical conditions for instability and design of the beam. A variable height problem has been formulated by means of a method described by Federhofer [7]. Numerous designs are obtained by use of Bessel equation analysis and a predicted optimum shape is forseen.

Throughout the discussion all problems are considered only for the elastic range of the material. Additionally,

the material is considered to be homogenous and isotropic. All problems analyzed are restricted to rectangular cross sections.

II. PROBLEMS OF OPTIMAL STRUCTURAL DESIGN

A. THE BASIC PROBLEM OF STRUCTURAL MECHANICS

The classical theory of structural mechanics is concerned with the behavior of continuous elastic bodies which are subjected to specific loading conditions. In general the components which define mechanical behavior are the displacement vector $\bar{u}$, and the stress and strain tensors, σ_{ij} and ε_{ij}.

The strain tensor is explicitly defined by the geometry of deformation of an elastic body. Hence, the strain-displacement relation may be represented by

$$\underline{\varepsilon} = \bar{\varepsilon}(\bar{u}) \tag{2.1}$$

From the constitutive laws of elasticity it is known that stress is related to strain, that is

$$\underline{\sigma} = \bar{\sigma}(\underline{\varepsilon}) \tag{2.2}$$

Thus, stress and strain are both representable as functions of the displacement. In classical theory any one of the three quantities, $\bar{u}$, $\bar{\sigma}$, and $\bar{\varepsilon}$ is sufficient for determining the other two.

The mechanical state of a structure is defined by a boundary value problem which is comprised of equilibrium equations and the associated boundary conditions. Thus, a classical problem in structural mechanics consists of using the constitutive relations between stress, strain and

displacement to derive and solve the governing equilibrium equations and their associated boundary conditions.

B. CALCULUS OF VARIATIONS

Problems concerned with the determination of extreme values of integrals whose integrands contain unknown functions below to that field known as the "calculus of variations" [8]. Such an extremal refers to the maximum, minimum or stationary value of a functional. A functional is a real valued function of a function.

Only the integral functional

$$I[f(\bar{x})] = \int_D F[f(\bar{x})]d\bar{x} \qquad (2.3)$$

will be considered herein, where D is the n-dimensional domain over which the n-dimensional vector $\bar{x}$ ranges. The function which extremizes $I[f(\bar{x})]$ is denoted as $f^*(\bar{x})$ and the associated value of the functional is I^* (i.e. $I^* = I[f^*(x)]$).

In general, problems in structural mechanics concerning optimization will assume the form

$$\int_{R_1} F(x;u,u',u'',\phi,\phi',\ldots)dx \qquad (2.4)$$

for a one dimensional case. A one dimensional case is generally characterized as having one dimension, its length, much larger than the other dimensions. In the above form x represents an independent variable. The quantities u, ϕ and their respective derivatives represent dependent variables. In this thesis u and ϕ represent displacement functions or

state variables. The nature of the specific problem governs the number of dependent and independent variables.

Performing a variation with respect to a functional is denoted by the symbol δ. From [8] it can be shown that a stationary value of the functional occurs for $\delta F = 0$. This operation yields what is known as a set of Euler equations. For example, if equation (2.4) is represented by the particular form

$$I = \int_D F(x;u,u',u'',\phi,\phi',A)dx \qquad (2.5)$$

then performing the extremal operation yields three Euler equations [9].

$$\delta_u F = 0: \quad \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u'}\right) + \frac{\partial}{\partial x^2}\left(\frac{\partial F}{\partial u''}\right) = 0 \qquad (2.6)$$

$$\delta_\phi F = 0: \quad \frac{\partial F}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \phi'}\right) = 0 \qquad (2.7)$$

$$_A F = 0: \quad \frac{\partial F}{\partial A} = 0 \qquad (2.8)$$

plus a set of natural boundary conditions.

C. THE ISOPERIMETRIC PROBLEM

The isoperimetric problem in the calculus of variations provides for extremizing functionals whose dependent variables are constrained by an integral relation. The integral constraint is accommodated by constant Lagrangian multipliers.

As an example, suppose that the maximum (or minimum) value of an integral

$$I = \int_R F(x;u,u')dx \qquad (2.9)$$

is sought, subject to the condition that another integral

$$J = \int_R G(x;u,u')dx \qquad (2.10)$$

is known to have a constant value. Using the method of Lagrange multipliers [8] an augmented functional

$$I+\lambda J = \int_R [F(x;u,u') + G(x;u,u')]dx \qquad (2.11)$$

may be constructed and its free extremum considered. The necessary condition for a stationary value is the Euler equation

$$\frac{\partial(F+ G)}{\partial u} - \frac{\partial}{\partial x} \frac{\partial(F+ G)}{\partial u'} = 0 \qquad (2.12)$$

where F has been replaced by $F^* = (F+\lambda G)$.

D. POTENTIAL ENERGY

The total potential energy of a conservative elastic system can be expressed as

$$T = U - V \quad . \qquad (2.13)$$

U represents the total strain energy of the structure, and V is the potential energy of external forces associated with the displacement of the system under any set of given loading conditions. The total strain energy of the system is the volume integral

$$U = \frac{1}{2} \int_V \sigma_{ij}\varepsilon_{ij} dV \qquad (2.14)$$

where σ_{ij} and ε_{ij} are components of the stress and strain tensors.

The potential energy of external forces is defined as the inner product of the external force vector $\bar{P}$ and the associated surface displacement vector $\bar{u}$, that is,

$$V = \int_s \bar{P} \cdot \bar{u} \, ds$$

where s is the domain over which $\bar{P}$ is defined.

E. PRINCIPLE OF MINIMUM POTENTIAL ENERGY

A powerful tool in the field of structural mechanics is the principle of minimum potential energy. This theorem states that the total potential energy of a conservative elastic system attains a minimum, with respect to variation of kinematically admissible displacement functions, when the displacement function satisfies equilibrium.

The total potential energy has previously been defined in terms of strain energy and external work. These quantities may be expressed as functions of the displacement vector $\bar{u}$. Thus, the equilibrium equations and associated boundary conditions are derivable from a variation with respect to the displacements of a potential energy functional.

The variational operation

$$\delta_u (U-V) = 0 \qquad (2.16)$$

can then be performed for which the resulting Euler equations physically represent the system equilibrium equations. The optimization method considered in the thesis is based on this important principle.

F. A STRUCTURAL OPTIMIZATION PROBLEM

As an example of structural optimization, consider the problem of obtaining the optimum shape for column buckling. The problem may be characterized by the statement: "For a column of given length and volume of material, determine the column shape for which the Euler buckling load is a maximum."

The column shape is considered to be such that all cross sections are similar. The moment of inertia I(x) is assumed to be related to area, A(x), that is

$$EI(x) = kA^n(x) \tag{2.17}$$

where E represents Young's modulus and k is a constant depending on the section considered. For example, the moments of inertia of rectangular sections of variable-width-constant height, and variable height-constant width are (Fig. 1),

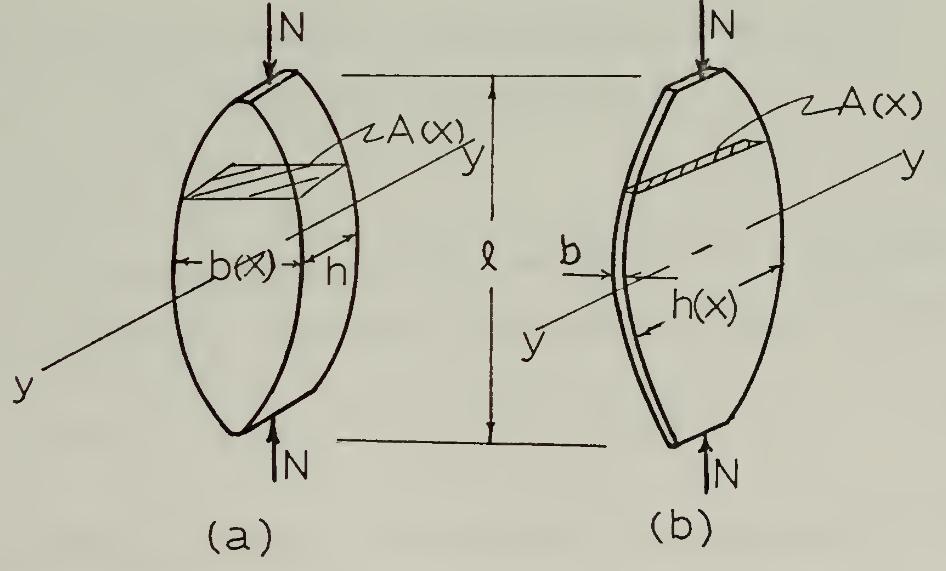

Figure 1. Column Shapes Analyzed, (a) variable width - constant height, (b) variable height - constant width.

$$EI_{yy} = \frac{Eh^3 b(x)}{12} = kA(x) \qquad (n = 1) \tag{2.18}$$

for case (a) and

$$EI_{yy} = \frac{Ebh^3(x)}{12} = kA^3(x) \qquad (n = 3) \tag{2.19}$$

for case (b).

For the Euler buckling problem the total potential energy is

$$T = \frac{1}{2} \int_0^{\ell} (EIv''^2 - Nv'^2)dx \tag{2.20}$$

where v represents lateral displacement for the buckled state. Introducing an isoperimetric condition for a constant volume of material V_o gives

$$\int_0^{\ell} A(x)dx = V_o \tag{2.21}$$

The volume constraint is accommodated through the formation of the functional T*, defined by

$$T^* = T - \lambda V_o \tag{2.22}$$

where λ is a constant Lagrangian multiplier. Combining (2.17), (2.20) and (2.22) the constrained functional is

$$T^* = \int_0^{\ell} (\frac{k}{2} A^n v''^2 - \frac{N}{2} v'^2 - \lambda A)dx . \tag{2.23}$$

The governing equations, in addition to the isoperimetric equation, are obtained by performing a variation with respect to v and A to obtain

$$\delta_v T^* = 0: \quad (kA^n v'')'' + Nv'' = 0 \qquad (2.24)$$

$$\delta_A T^* = 0: \quad \tfrac{1}{2} nkA^{n-1} \, v''^2 - \lambda = 0 \qquad (2.25)$$

Equation (2.24) is the equilibrium equation and (2.25) an optimality condition.

The non-linear equations (2.24), (2.25) and (2.21) define the optimum design with respect to constant volume - maximum load. In general these are extremely difficult to solve in closed form. However, an exact solution can be obtained for case (a) where n = 1. These governing equations then become

$$(kAv'')'' - Nv'' = 0 \qquad (2.26)$$

$$\frac{1}{2} kv''^2 = \lambda \qquad (2.27)$$

$$\int_0^{\ell} A(x)dx = V_o \qquad (2.21)$$

Equation (2.27) implies constant bending curvature, i.e.

$$v'' = \sqrt{2\lambda/k} \qquad (2.28)$$

Substituting (2.28) into (2.26) and integrating twice gives

$$A = \frac{N}{k} \left(\frac{x^2}{2} + C_1 x + C_2\right) \qquad (2.29)$$

Similarly

$$v = \sqrt{2\lambda/k} \left(\frac{x^2}{2} + C_3 x + C_4\right) . \qquad (2.30)$$

If the column is considered to be simply supported, the boundary conditions for displacement are

$$v(0) = v(\ell) = 0 \qquad (2.31)$$

for which $C_3 = -\frac{\ell}{2}$, $C_4 = 0$, and

$$v(x) = \sqrt{2\lambda/k}\ \left(\frac{x^2}{2} - \frac{\ell x}{2}\right) . \tag{2.32}$$

The moment conditions are

$$M(o) = M(\ell) = 0 \tag{2.33}$$

from which $C_1 = -\frac{\ell}{2}$, $C_2 = 0$, and

$$A(x) = \frac{N}{k}\left(\frac{x^2}{2} - \frac{\ell x}{2}\right) . \tag{2.34}$$

The critical load is now determined from the isoperimetric condition

$$\int_0^{\ell} \frac{N}{k}\left(\frac{x^2}{2} - \frac{\ell x}{2}\right) = V_o \tag{2.35}$$

for which

$$N_{optimum} = \frac{-12kV_o}{\ell^3} = \frac{-Eh^2V_o}{\ell^3} . \tag{2.36}$$

Substituting this expression into (2.34) determines the optimum design in terms of known constants, that is

$$A(x) = \frac{V_o}{12\ell^3}\ (\ell x - x^2) . \tag{2.37}$$

Note that the Lagrange multiplier does not appear in either of the expressions for A(x) or N. This is in distinction to strength optimization problems where λ is a representation of maximum strength and remains in the final analysis (see Appendix A).

III. THE LATERAL BUCKLING PROBLEM

A. GENERAL DESCRIPTION

The problem of lateral buckling is important in the design of beams which have no lateral support and possess a shape such that the beam height is large in comparison to the width. Such a configuration may become laterally unstable when the loading reaches a critical value. The resulting failure or "buckle" is a combination of lateral bending and twisting away from the horizontal axis (Fig. 2). Prior to buckling the load needed to cause this failure may be considerably less than the corresponding load that would

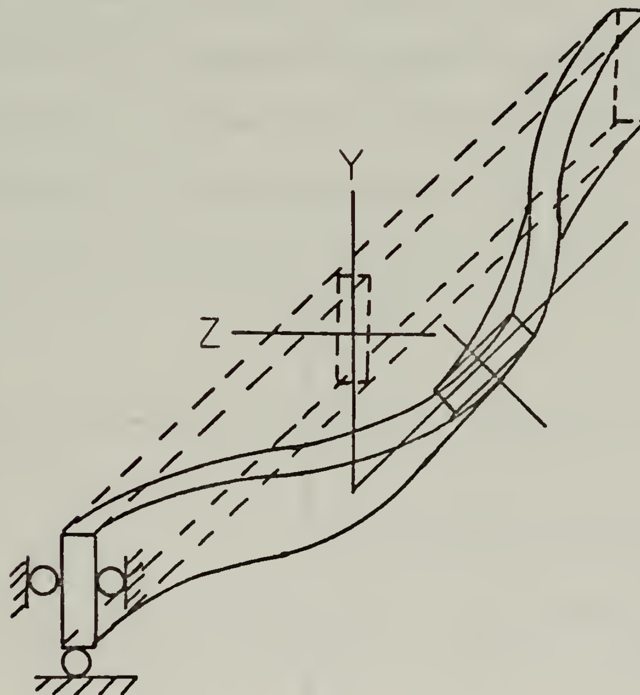

Figure 2. Simply Supported Beam in a Lateral Buckled State.

cause bending failure. This is due to a large flexural rigidity of the beam in the vertical plane of bending as compared to a much smaller lateral bending resistance in the plane perpendicular to the applied load.

A brief but informative discussion of the historical development of the lateral buckling problem is given by Bleich [12] and will not be included here. The methods used for development of the problem in this thesis are similar to those used by Sechler [13] and Timoshenko [14].

B. DEVELOPMENT OF CLASSICAL SOLUTION

While the solution to the classical lateral buckling problem may be found in a number of texts, its inclusion herein is considered necessary for understanding the descriptive geometry of the problem and later development of the problem of a variable height solution. The example to be considered is that of a simply supported beam loaded at its centroid by a concentrated load P at the origin (Fig. 3). The centroidal loading condition is necessary to

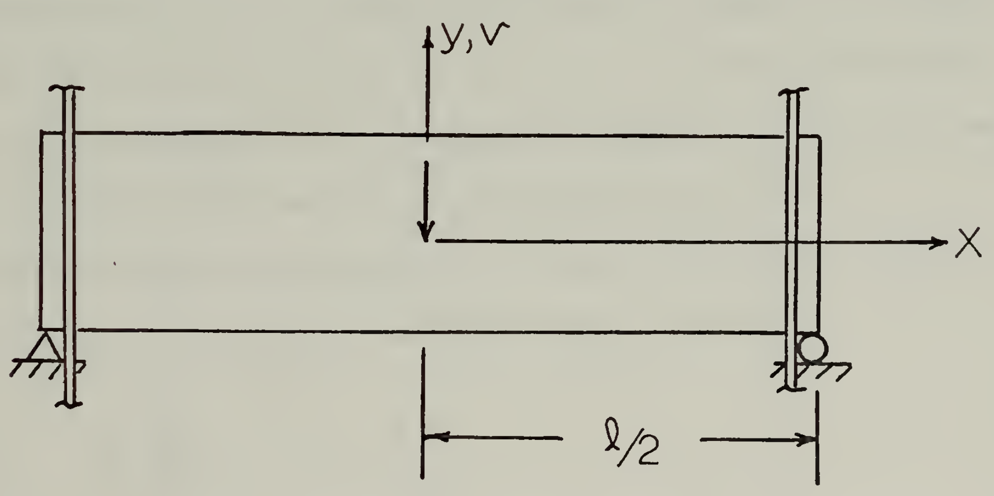

Figure 3. Simply Supported Beam Loaded at the Centroid.

simplify the problem. Loading at any point other than the centroid introduces an additional work term which leads to some mathematical difficulties. This condition is described in Appendix B.

It is assumed that the load P remains in the vertical xy plane throughout buckling. The ends of the beam are so constrained as to prevent rotation in the yz plane. The buckled configuration is shown in Figure 4.

The deflection of the beam is defined at any point x from the origin by the components u, v and ϕ. Components u and v represent the displacement of the centroid in the z and y directions while ϕ is the angle of rotation of the cross section in the yz plane. Thus, the boundary conditions for the problem are

$$u(\ell/2) = \phi(\ell/2) = 0 \tag{3.1}$$

$$u'(0) = \phi'(0) = 0 \tag{3.2}$$

The fixed coordinate axes are labeled x, y and z. In the buckled state the principle axes of symmetry at any section mn are defined as ξ, η, and ζ. The usual right hand rule convention for vectors is used to describe positive moments with respect to the described axes.

By the usual assumptions of small deflection theory the equations for principle moments can be written

$$EI_\zeta \frac{d^2\eta}{d\xi^2} \simeq EI_\zeta v'' = M_\zeta \tag{3.3a}$$

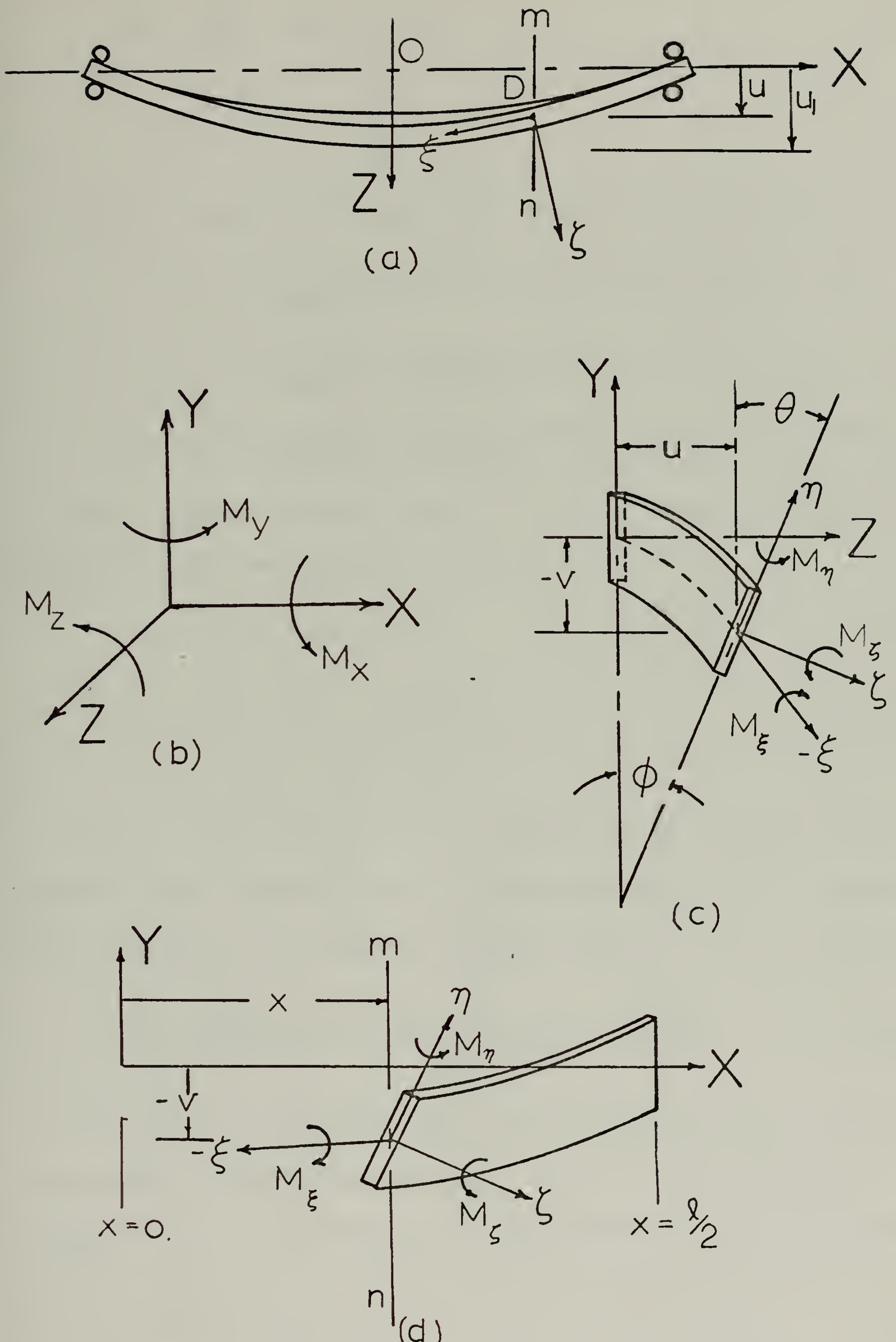

Figure 4. Lateral Buckling Configuration.

$$EI_\eta \frac{d^2\zeta}{d\xi^2} \simeq EI_\eta u'' = -M_\eta \tag{3.3b}$$

$$GJ \frac{d\theta}{d\xi} \simeq GJ\phi' = M_\xi \tag{3.3c}$$

These moments are physically described as follows:

i) M_ζ is the usual bending moment due to the load P

ii) M_η is the bending moment associated with lateral buckling

iii) M_ξ is the twisting moment associated with lateral buckling.

With respect to the fixed x, y and z axes the above moments can be represented [13] as

$$M_\xi = M_x + M_z u' = GJ\phi' \tag{3.4a}$$

$$M_\eta = M_y + M_z\phi = -EI_\eta u'' \tag{3.4b}$$

$$M_\zeta = -M_y\phi + M_z = EI_\zeta v'' \tag{3.4c}$$

Let $C_1 = GJ$ and $C_2 = EI_\eta$. Differentiating, the first equation with respect to x and substituting for u" from the second results in a general equation which can be written

$$\phi'' + \frac{M_z^2}{C_1C_2}\phi = \frac{1}{C_1}M_x' + \frac{1}{C_1}M_z'u' - \frac{M_zM_y}{C_1C_2} . \tag{3.5}$$

For the problem considered let the deflection at the beam center (maximum deflection) in the z direction be denoted by u_1. The moments at any distance x are determined as

$$M_x = -P/2\ (u_1-u) \tag{3.6a}$$

$$M_y = 0 \tag{3.6b}$$

$$M_z = P/2\ (\ell/2-x) \tag{3.6c}$$

Differentiating and substituting into equation (3.5) yields

$$\phi'' + \frac{P^2(\frac{\ell}{2} - x)^2}{4C_1C_2}\,\phi = 0\ . \tag{3.7}$$

Introducing the new variable $\beta = (\ell/2-x)$ and letting $k^2 = \frac{P^2}{4C_1C_2}$ equation (3.7) becomes

$$\phi'' + k^2\beta^2\phi = 0\ . \tag{3.8}$$

Equation (3.8) is now in the form of a Bessel equation for which the general solution has the form

$$\phi = \beta^{\frac{1}{2}}\,[a_1 J_{\frac{1}{4}}(\tfrac{k}{2}\,\beta^2) + a_2 J_{-\frac{1}{4}}\,(\tfrac{k}{2}\,\beta^2)] \tag{3.9}$$

where $J_{\frac{1}{4}}$ and $J_{-\frac{1}{4}}$ are Bessel functions of the first kind of orders $\frac{1}{4}$ and $-\frac{1}{4}$ respectively [14]. For the simply supported beam the boundary conditions with respect to ϕ are

$$\phi = 0 \quad \text{at} \quad \beta = 0 \qquad (\text{i.e. } x = \ell/2) \tag{3.10a}$$

$$\phi' = 0 \quad \text{at} \quad \beta = \ell/2 \qquad (\text{i.e. } x = 0)\ . \tag{3.10b}$$

The first boundary condition yields $a_2 = 0$. Differentiation of ϕ with respect to β now yields

$$\frac{d\phi}{d\beta} = a_1 k\beta^{3/2}\ J_{-3/4}(\tfrac{k}{2}\,\beta^2)\ . \tag{3.11}$$

Applying the second boundary condition shows that buckling occurs when

$$J_{-3/4}\ (\frac{k\ell^2}{8}) = 0\ . \tag{3.12}$$

From a table of Bessel functions [15] the first root of (3.12) is

$$\frac{k\ell^2}{8} = 1.058508 \ . \qquad (3.13)$$

Substituting for K, the buckling load is found to be

$$P_{cr} = \frac{16.936\sqrt{C_1 C_2}}{\ell^2} \qquad (3.14)$$

Equation (3.14) represents the exact solution for the problem considered and will be used as a comparative value for critical loads calculated for variable cross sectional designs later.

C. OPTIMIZATION TECHNIQUE APPLIED TO LATERAL BUCKLING

The energy formulation described in Section II is now used to develop the governing equations for an optimum shape of a thin beam which undergoes lateral buckling. The assumptions made during the preceeding section apply. Additional assumptions include neglecting the strain energy due to "warping" [14] and elongation of the beam.

The case considered is that of a variable height-constant width beam (Fig. 5). The principle moments of inertia with respect to the buckled state are

$$I_\eta = \frac{1}{12} b_o^3 h(x) = \frac{b_o^2}{12} A(x) \qquad (3.15)$$

$$J = \frac{1}{3} b_o^3 h(x) = \frac{b_o^2}{3} A(x) \ . \qquad (3.16)$$

The strain energy due to lateral bending may be written

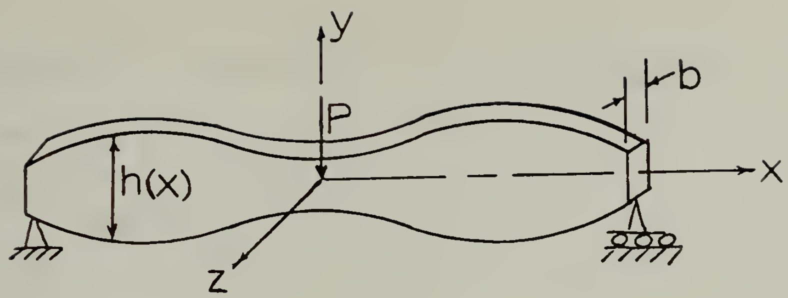

Figure 5. Variable Height, Constant Width Beam.

$$U_1 = \int_0^{\ell/2} C_1 A u''^2 \, dx \qquad (3.17)$$

where $C_1 = E b_o^2/12$. The strain energy associated with twisting of the structure is

$$U_2 = \int_0^{\ell/2} C_2 A \phi'^2 \, dx \qquad (3.18)$$

where $C_2 = G b_o^2/3$.

As the beam buckles the load P is lowered an amount which may be represented as [14]

$$\phi u'' \, (\ell/2 - x) \, dx \, . \qquad (3.19)$$

The potential energy of external forces associated with the buckling configuration is determined by

$$V = P \int_0^{\ell/2} \phi u'' \, (\ell/2 - x) \, dx \, . \qquad (3.20)$$

The total potential energy of the system is U - V or

$$T = \int_0^{\ell/2} [C_1 A u''^2 + C_2 A \phi'^2 - P \phi u'' \, (\ell/2 - x)] dx . \qquad (3.21)$$

To formulate a functional by the method described earlier,

an isoperimetric constraint is introduced for a given volume of material,

$$\int_0^{\ell/2} A dx = V_o \quad . \tag{3.22}$$

The augmented functional may be written

$$T^* = \int_0^{\ell/2} [C_1 A u''^2 + C_2 A \phi'^2 - P\phi u''(\frac{\ell}{2} - x) - \lambda A] dx. \tag{3.23}$$

Performing a variation with respect to the variables u, ϕ and A yields three Euler equations:

$$\delta_u T^* = 0: \quad [2C_1 A u'' - P\phi(\frac{\ell}{2} - x)]'' = 0 \tag{3.24}$$

$$\delta_\phi T^* = 0: \quad 2C_2(A\phi')' + Pu''(\frac{\ell}{2} - x) = 0 \tag{3.25}$$

$$\delta_A T^* = 0: \quad C_1 u''^2 + C_2 \phi'^2 - \lambda = 0 \ . \tag{3.26}$$

Note that the variation with respect to the state variables u and ϕ yields equilibrium equations. Variation with respect to the design variable A introduces an optimality equation. The associated boundary conditions are:

$$u(\frac{\ell}{2}) = \phi(\frac{\ell}{2}) = 0 \tag{3.27}$$

$$u'(0) = \phi'(0) = 0 \tag{3.28}$$

Integration of equation (3.24) twice and evaluating at the boundary conditions determines

$$2C_1 A u'' - P\phi(\frac{\ell}{2} - x) = 0 \tag{3.29}$$

Integration of (3.25) yields the result

$$C_2 A\phi' + \frac{P}{2} u' (\frac{\ell}{2} - x) + \frac{P}{2} (u_1 - u) = 0 \tag{3.30}$$

where u_1 represents the maximum deflection at the beam midspan.

The above equilibrium and optimality equations along with the isoperimetric condition (3.22) define the optimum shape for the structure. These governing equations are noted to be non-linear and an attempt to solve them has been unsuccessful. Unfortunately, these equations resist uncoupling, which precludes getting to the isoperimetric condition as was accomplished easily in Section III. Additionally, no computer techniques were found which could solve systems of simultaneous nonlinear equations.

After considerable efforts to solve the above non-linear system proved fruitless, an attempt to formulate a variable height solution from a specified polynomial was undertaken in hopes of predicting or converging to an optimum solution. The method used and results obtained are included in the following section.

IV. A VARIABLE HEIGHT BUCKLING PROBLEM

A. DESCRIPTION OF THE PROBLEM

The problem of lateral buckling of a variable height, constant width beam is now considered. Again, the problem to be investigated is that of a simply supported beam, loaded at the midspan centroid. However, the height now varies according to the power law

$$h = h_o(1 - \frac{2x}{\ell})^n \qquad (4.1)$$

where h_o is the height at the center section of the beam (Fig. 6). The value of n is restricted to lie between the interval $0 \leq n \leq 1$ for reasons which become physically obvious in later analysis of the problem. The idea for using this type of formulation was based on the results

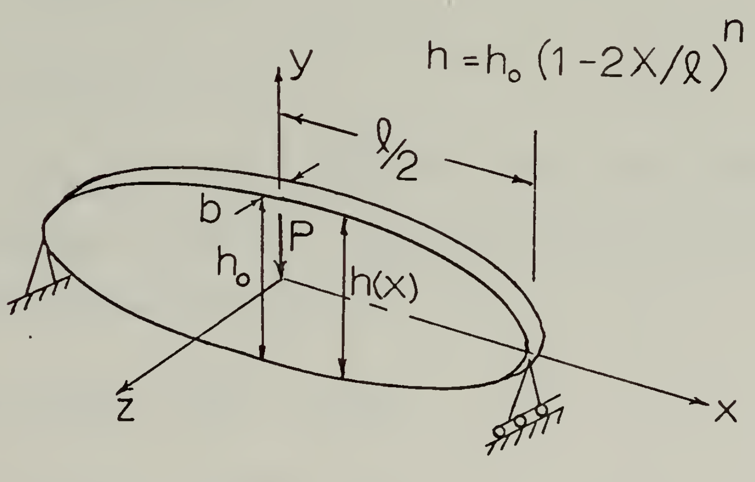

Figure 6. Variable Height Beam Based on the Power Law $h_o(1 - 2x/\ell)^n$.

obtained for a variable height cantilever problem by Federhofer [7].

Differentiating equation (3.4a) with respect to x and noting that the previous constants C_1 and C_2 are now variable functions of x, reduces (3.4a) to the form

$$C_1\phi'' + C_1'\phi' = M_x' + M_z'u' + M_zu'' \quad . \tag{4.2}$$

Substituting (3.4b) into (4.2) for u" and recalling the moment conditions from (3.6) (and their respective derivatives) reduces (4.2) to

$$\phi'' + \frac{1}{C_1} C_1'\phi' + \frac{P^2(\frac{\ell}{2} - x)^2}{4C_1C_2} \phi = 0 \quad . \tag{4.3}$$

Letting

$$\gamma = (1 - \frac{2x}{\ell}) \tag{4.4}$$

and using the relations

$$C_1 = \frac{1}{12} Eb^3 \ h(x)$$

$$C_2 = \frac{1}{3} Gb^3 \ h(x)$$

the lateral and flexural stiffness at the beam center may be designated as C_1^o and C_2^o. Thus, the new variable stiffnesses can be represented by

$$C_1 = C_1^o \ (1 - \frac{2x}{\ell})^n = C_1^o\gamma^n \tag{4.5a}$$

$$C_2 = C_2^o \ (1 - \frac{2x}{\ell})^n = C_2^o\gamma^n \quad . \tag{4.5b}$$

Using the chain rule

$$\frac{d\phi}{d\gamma} = \frac{d\phi}{dx} \frac{dx}{d\gamma} \tag{4.6}$$

and noting

$$x = \frac{\ell}{2} (1-\gamma)$$

$$\phi'' = \frac{d^2\phi}{dx^2} = \frac{4}{\ell^2} \frac{d^2\phi}{d\gamma^2} \tag{4.7}$$

Similarly

$$\frac{1}{C_1} \frac{dC_1}{dx} \frac{d\phi}{dx} = \frac{4}{\ell^2} \frac{n}{\gamma} \frac{d\phi}{d\gamma} \tag{4.8}$$

Substitution of (4.7) and (4.8) into equation (4.3) yields

$$\frac{d^2\phi}{d\gamma^2} + \frac{n}{\gamma} \frac{d\phi}{d\gamma} + \frac{P^2\ell^4\gamma^{2(1-n)}}{64C_1^oC_2^o} = 0 . \tag{4.9}$$

The dimensionless parameter K^2 may be defined

$$\frac{P^2\ell^4}{64C_1^oC_2^o} = K^2 . \tag{4.10}$$

Equation (4.9) may now be written

$$\frac{d^2\phi}{d\gamma^2} + \frac{n}{\gamma} \frac{d\phi}{d\gamma} + K^2\gamma^{2(1-n)}\phi = 0 . \tag{4.11}$$

Transformation of equation (4.11) into a Bessel equation can be accomplished by the following substitutions.

$$\rho = \frac{K}{2-n} \gamma^{2-n} \tag{4.12a}$$

$$\phi = \gamma^{\frac{1-n}{2}} \beta(\gamma) \tag{4.12b}$$

Equation (4.11) is then reduced to the form

$$\frac{d^2\beta}{d\rho^2} + \frac{1}{\rho}\frac{d\beta}{d\rho} + (1 - \frac{\nu^2}{\rho^2}) = 0 \; . \qquad (4.13)$$

where ν is representative of

$$\nu = \frac{1-n}{2(2-n)} \; . \qquad (4.13a)$$

Verification of the transformation is accomplished in Appendix C.

1. Non-integer Value of ν

For the non-integer values of the parameter ν between zero and one equation (4.13) has the following solution [16]

$$\beta = \alpha_1 J_\nu(\rho) + \alpha_2 J_{-\nu}(\rho) \qquad (4.14)$$

where J_ν and $J_{-\nu}$ are Bessel functions of the first kind and order ν.

Recalling (4.12b), (i.e. $\phi = \gamma^{\frac{1-n}{2}} \beta(\gamma)$), the complete solution is determined by

$$\phi = \gamma^{\frac{1-n}{2}} [\alpha_1 J_\nu(\rho) + \alpha_2 J_{-\nu}(\rho)] \qquad (4.15)$$

The boundary conditions associated with the lateral buckling problem are:

$$\phi(\gamma = 0) = 0 \qquad (\text{i.e. } x = \frac{\ell}{2}) \qquad (4.16a)$$

$$\frac{d\phi}{d\gamma}(\gamma = 1) = 0 \qquad (\text{i.e. } x = 0) \; . \qquad (4.16b)$$

From the first condition and expansion of $J_{-\nu}$, it is concluded that $\alpha_2 = 0$. Therefore (4.15) becomes

$$\phi = \gamma^{\frac{1-n}{2}} [\alpha_1 J_\nu(\rho)] \qquad (4.17)$$

for which differentiation yields

$$\frac{d\phi}{d\gamma} = \alpha_1 \left[\frac{1-n}{2} \gamma^{\frac{-1-n}{2}} J_\nu(\rho) + \gamma^{\frac{1-n}{2}} J'_\nu(\rho) \frac{d\rho}{d\gamma} \right] . \tag{4.18}$$

It is additionally known that

$$\frac{d\rho}{d\gamma} = K\gamma^{1-n} \tag{4.19}$$

and [17]

$$J'_\nu(\rho) = - J_{\nu+1}(\rho) + \frac{\nu}{\rho} J_{\nu}(\rho) \; ^{*} \tag{4.20}$$

Evaluation of $\frac{d\phi}{d\gamma}$ at the second boundary condition with $\gamma = 1$ and $\rho = K/(2-n)$ yields the solution

$$J_\nu \left(\frac{K}{2-n}\right) = \frac{K}{1-n} J_{\nu+1} \left(\frac{K}{2-n}\right). \tag{4.21}$$

The buckling loads may now be determined by solution of equation (4.21). This may be accomplished by independently calculating each case

$$J_\nu \left(\frac{K}{2-n}\right) = 0 \tag{4.21a}$$

and

$$\frac{K}{1-n} J_{\nu+1} \left(\frac{K}{2-n}\right) = 0 \tag{4.21b}$$

and determining the initial intersection of the resulting Bessel equation curves. For each considered value of n, K is found numerically from the argument of J. A computer

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*Handbook of Mathematical Functions, National Bureau of Standards Applied Mathematics, Series 55, p. 363, June 1964.

solution for this problem is included in the Computer Program section of the thesis. The Bessel equation curves which show the intersecting roots are included in Appendix D.

When n=0 the beam design reduces to that of the rectangular case. The computed solution from equation (4.21a) and (4.21b) provides an argument value of 1.05851. Thus, the critical load is determined by

$$\frac{K}{2-n} = 1.05851 \tag{4.22}$$

or

$$P_{cr} = \frac{16.936\sqrt{C_1^o C_2^o}}{\ell^2} \tag{4.23}$$

precisely the value determined previously. The values for other solutions are discussed in a later section.

2. Integer Value of ν

When the value of n is equal to one the resulting value of ν becomes zero and equation (4.13) is reduced to the form

$$\frac{d^2\beta}{d\rho^2} + \frac{1}{\rho}\frac{d\beta}{d\rho} + \beta = 0 \,. \tag{4.24}$$

The complete solution of this equation can be shown to be of the form [17]

$$\beta = \alpha_1 J_o(\rho) + \alpha_2 Y_o(\rho) \tag{4.25}$$

where Y_o is known as a Weber function or Bessel function of the second kind. Substituting for ϕ, the above equation is transformed to

$$\phi = \gamma^{\frac{1-n}{2}} [\alpha_1 J_o(\rho) + \alpha_2 Y_o(\rho)] . \qquad (4.26)$$

This solution is valid only if ν is of an integer order. Thus, in the restricted range n is allowed only to be equal to one, and the γ multiplier must be to the zero power. Equation (4.24) reduces to

$$\phi = \alpha_1 J_o(K\gamma) + \alpha_2 Y_o(K\gamma) . \qquad (4.27)$$

The value of $Y_o(K\gamma)$ for $\gamma = 0$ (i.e. $x = \frac{\ell}{2}$) is $-\infty$ which implies that α_2 is equal to zero. Differentiating ϕ with respect to the remaining term and evaluating at the second boundary condition ($\gamma = 1$, $x = 0$) requires that

$$-J_1(K) = 0 \qquad (4.28)$$

for the solution to (4.26). This value is found [18] to be either the trivial case (i.e. zero) or 3.8317. The latter result implies a singularity condition with respect to the previous non-integer solutions. A graphical representation (Fig. 7) of values of n versus the roots of equation (4.21) does, in fact, imply the resulting value converges to zero as n approaches one as a limit. Therefore, the trivial case is the solution. The beam is apparently unstable for any load when in this configuration.

B. CALCULATION OF AMENDED LOADS

Extension of the problem is now made in an attempt to find the design among all those analyzed that maximizes the buckling load. Initially h_o was described as a fixed height at the beam midspan. This center height is now allowed to

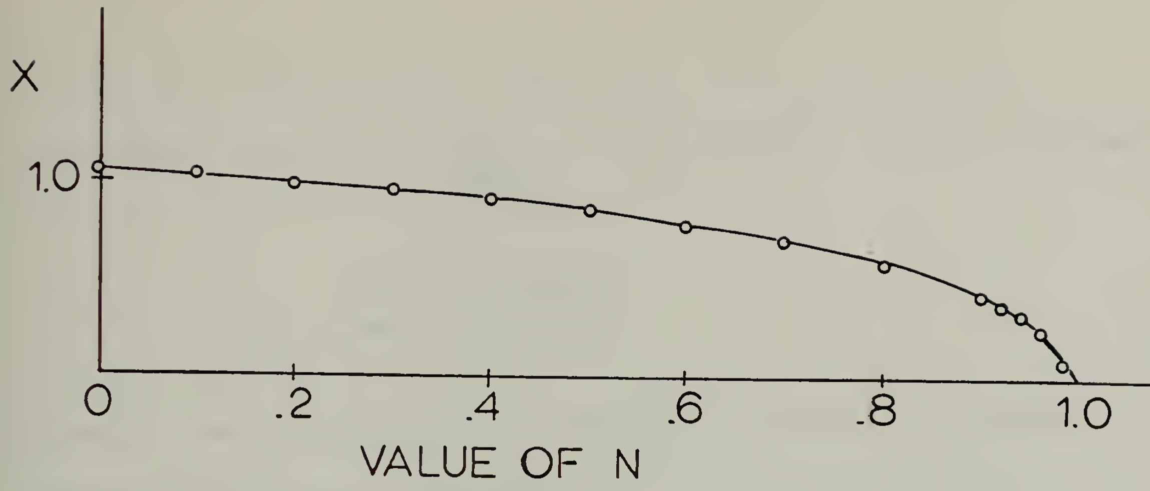


Figure 7. Convergence of the Root x as N approaches One.

vary such that each variable design studied will have the same volume as the constant rectangular beam.

This may be accomplished by assuming a constant width for all designs, integrating the power law (4.1) and setting the resulting value equal to that of the classical rectangular beam. Using symmetry and integrating the power law from x=0 to $x = \frac{\ell}{2}$

$$b \int_0^{\ell/2} h_o \left(1 - \frac{2x}{\ell}\right)^n dx = V_o \qquad (4.29)$$

determines the volume of the variable design. For the rectangular beam the volume for one-half the structure is $\frac{h_r bd}{2}$. Thus, equating the two determines

$$h_o = (n+1)\, h_r \qquad (4.30)$$

It is obvious that the center height must be expanded for the variable design in order that the volumes are compatible.

When h_o is restricted to a fixed value the critical loads may be calculated as described in Section IV-A, i.e.

using the intersecting value of (4.21a) and (4.21b).

Recalling that $K = \frac{P\ell^2}{8\sqrt{C_1^o C_2^o}}$, the critical load is defined as

$$P_{crit} = \frac{8(2-n)x\sqrt{C_1^o C_2^o}}{\ell^2} \quad (4.31)$$

where x represents the intersecting value of the Bessel equation curves. The amended load is now found by substituting for C_1^o, C_2^o and h_o.

$$P_{crit} = \frac{8(2-n)xh_o}{\ell^2}\sqrt{b^6\ EG/36} \quad (4.32)$$

Thus,

$$P^* = \frac{8(2-n)x(n+1)h_r}{\ell^2}\sqrt{C_3 C_4} \quad (4.33)$$

where

$$C_3 = \frac{b^6}{36} \quad (4.34)$$

$$C_4 = EG. \quad (4.35)$$

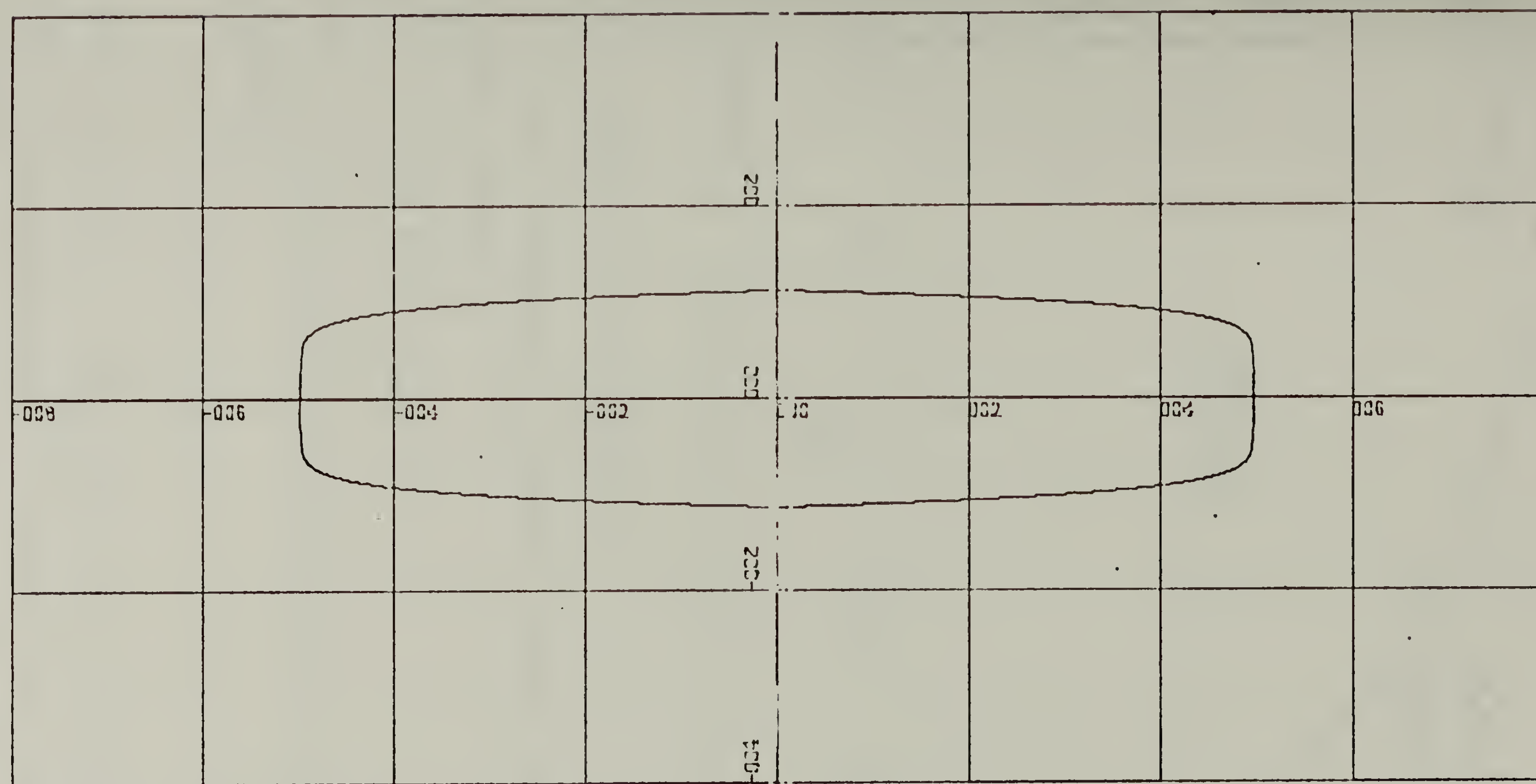
The amended load is referred to as P* since its value is greater than its unexpanded counterpart. It is left in the above form in order that it may be readily compared to the solution for the rectangular beam.

A computer solution is included in the Computer Program Section and provides for determining the intersecting value of (4.21a) and (4.21b). It additionally calculates the critical and amended loads for each value of considered n. These results are summarized in Table I below.

TABLE I

n	ν	ν+1	Pcrit	P*	P*/Pu
0	0.2500	1.2500	16.936=Pu	16.936	1.0
.1	0.2368	1.2368	15.618	17.179	1.014
.2	0.2222	1.2222	14.288	17.145	1.012
.3	0.2059	1.2059	12.944	16.827	0.994
.4	0.1875	1.1875	11.580	16.212	0.957
.5	0.1667	1.1667	10.189	15.283	0.902
.6	0.1429	1.1429	8.758	14.013	0.827
.7	0.1154	1.1154	7.263	12.347	0.729
.8	0.0833	1.0833	5.656	10.181	0.601
.9	0.0455	1.0455	3.794	7.210	0.426
0.98	0.0010	1.0010	0.507	1.013	0.059
1.0	0	--	0	0	0

The best design exists between n=0.1 and 0.2. Additional calculations indicated that n=0.14 offers the largest buckling load (P*=17.199) for the variable height formulation. This design is shown in Figure 8. Calculation of values for P*/Pu indicates that there is little change for the first six values of n. A graph at P* versus n is shown in Figure 9. Figures 27-38 in Appendix E illustrate the various designs studied in the interval of n=0 to n=1.



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VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.14

Figure 8. Simply Supported Beam Design -
Case $h = h_o(1- 2x/L)^{0.14}$ $P^* = 17.199$.

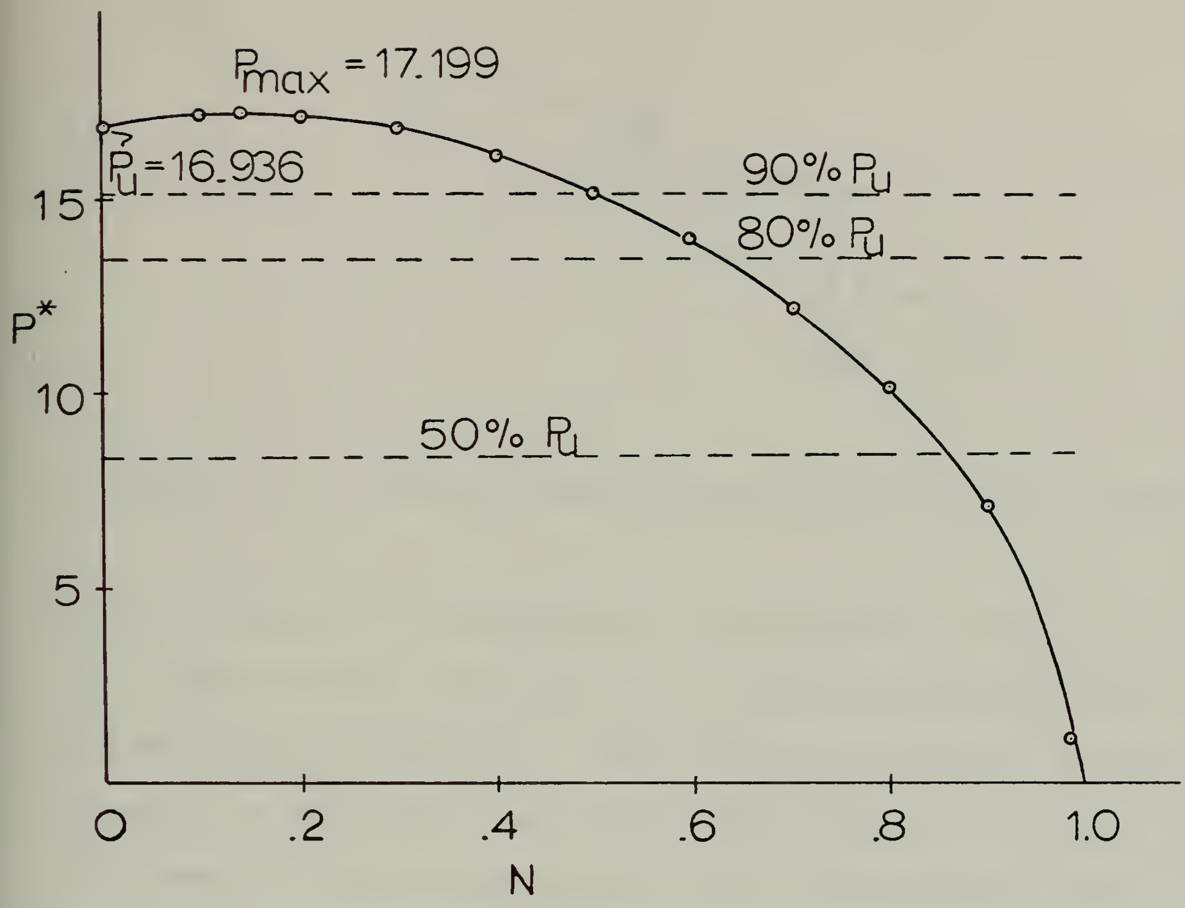

Figure 9. Amended Load P* Versus Values of N, (case 1).

C. EXTENSION OF THE PROBLEM

The formulation for design criteria was previously based on the assumption

$$h = h_o(1 - 2x/\ell)^n. \quad (4.1)$$

The case will now be considered for the design based on the condition

$$h = h_o(1 + 2x/\ell)^n. \quad (4.36)$$

This condition is physically shown in Figure 10. By inspection of the governing equations of the forementioned solution, it is evident that the sign change makes no difference in

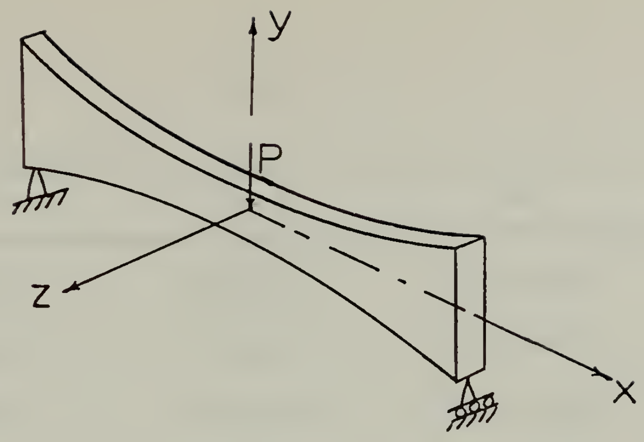

Figure 10. Variable Height Beam Based on $h = h_o(1+2x/\ell)^n$.

the solution of the problem. The identical critical loads are again found. However, each design is restricted to the same volume of material. This requires that the midspan height be decreased while the end heights of the beam are increased. This modification has a significant effect on the amended loads.

Integrating (4.36) and using the same technique described in Section IV-B, the center height reduces to

$$h_o \left[\frac{n+1}{(2)^{n+1} - 1}\right] h_r \tag{4.37}$$

while the ends expand to

$$h_e = \frac{2^{n+1}}{2(n+1)} h_r \tag{4.38}$$

A computer solution for this problem is included at the end of the thesis. The results are summarized in Table II.

The structure is less stable for any comparable condition listed in the previous section. Figures 39-48, Appendix E include designs analyzed for this problem. A

TABLE II

n	x	Pcrit	P*	% Decrease
0	1.05851	16.936	16.936	--
.1	1.02747	15.618	15.023	11.30
.2	0.99220	14.288	13.215	21.97
.3	0.95178	12.944	11.508	32.05
.4	0.90470	11.580	9.891	41.60
.5	0.84907	10.189	8.359	50.64
.6	0.78200	8.758	6.898	59.27
.7	0.69840	7.263	5.490	67.58
.8	0.58920	5.656	4.102	75.78
.9	0.43119	3.794	2.639	84.41

plot of the amended load P* versus n is shown in Figure 11. Note that all designs reflect a decrease in buckling load as compared to Figure 9 and Table I. Also, there is no range of designs for which the value P* remains relatively stationary.

D. LATERAL BUCKLING OF A CANTILEVER

A similar analysis [7] for a cantilever beam shows that the buckling loads are determined by

$$J_{-\nu}\left(\frac{K}{2-n}\right) = 0 \tag{4.39}$$

where K is defined as

$$K = \frac{P\ell^2}{\sqrt{C_1^o C_2^o}} \tag{4.40}$$

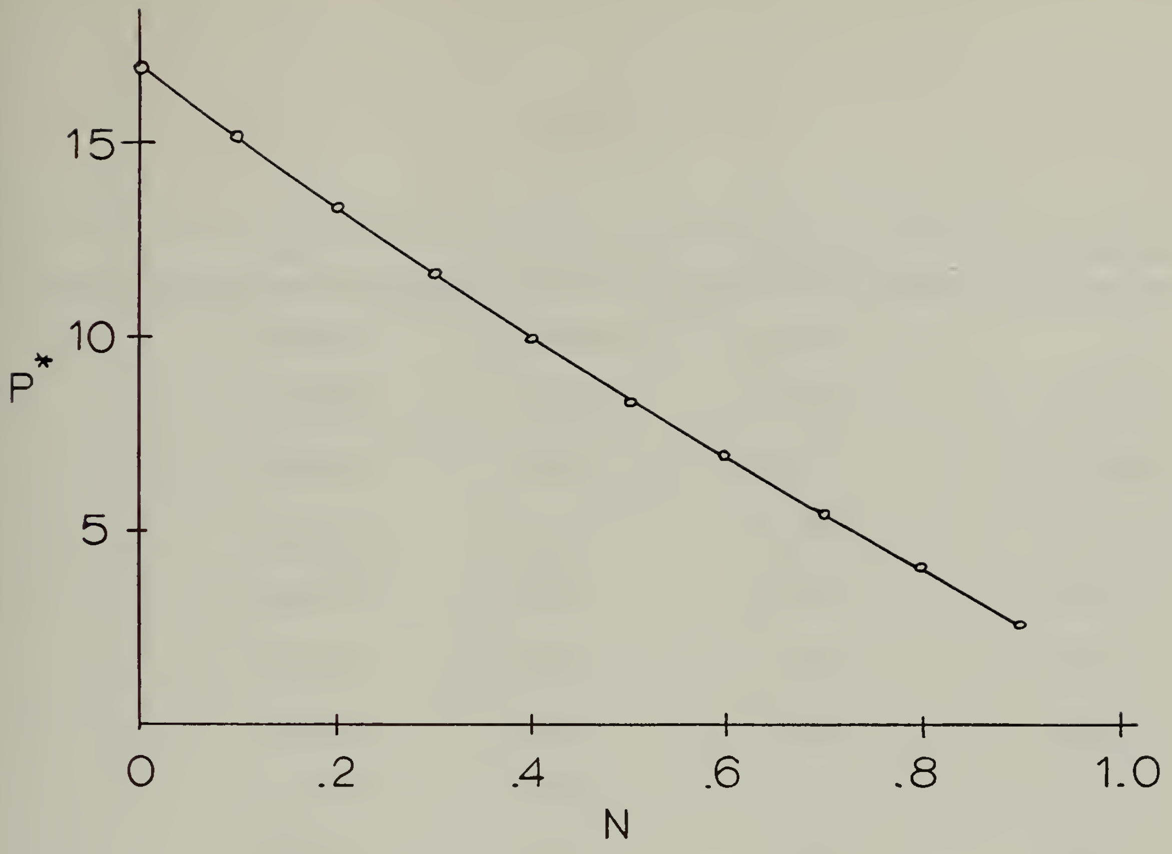

Figure 11. Amended Load P* Versus Values of N (case 2).

A computer solution was used to calculate the first zero of a negative order Bessel function of the first kind, and used this result to obtain the critical and amended loads for the structure. These results are summarized below in Table III. The last value is obtained by evaluation of the Bessel function of the second kind and use of tables [18]. The values of P* versus n are shown in Figure 12. Note in particular that all values are greater than the ultimate critical load, Pu, for the rectangular case.

The best design is determined for the case of n=0.75, and is noted as P*=4.918. This shape is shown in Figure 13. The range of shapes considered in the formulation are shown in Appendix E.

TABLE III

n	ν	Pcrit	P*	% Increase
0	-.2500	4.013=Pu	4.013	--
.1	-.2368	3.858	4.239	5.63
.2	-.2222	3.694	4.433	10.47
.3	-.2059	3.534	4.595	14.50
.4	-.1875	3.374	4.724	17.72
.5	-.1667	3.214	4.821	20.13
.6	-.1429	3.053	4.885	21.73
.7	-.1154	2.892	4.917	22.53
.8	-.0833	2.731	4.915	22.48
.9	-.0455	2.568	4.880	21.60
1.0	0	2.405	4.810	19.86

E. COMPARATIVE ANALYSIS OF SOLUTIONS

A comparative analysis between the solutions determined for the cantilever and simply supported beams provides some interesting insight into the actual physics of the problem. The results obtained from the cantilever solution appear to be a near optimum shape for the structure. At first sight analysis of the simply supported beam does not appear to yield the optimum shape, although a better design than that of classical solution is obtained. However, a close examination of the physics of the problem leads to a prediction for the possible optimum shape, although it was not analytically determined.

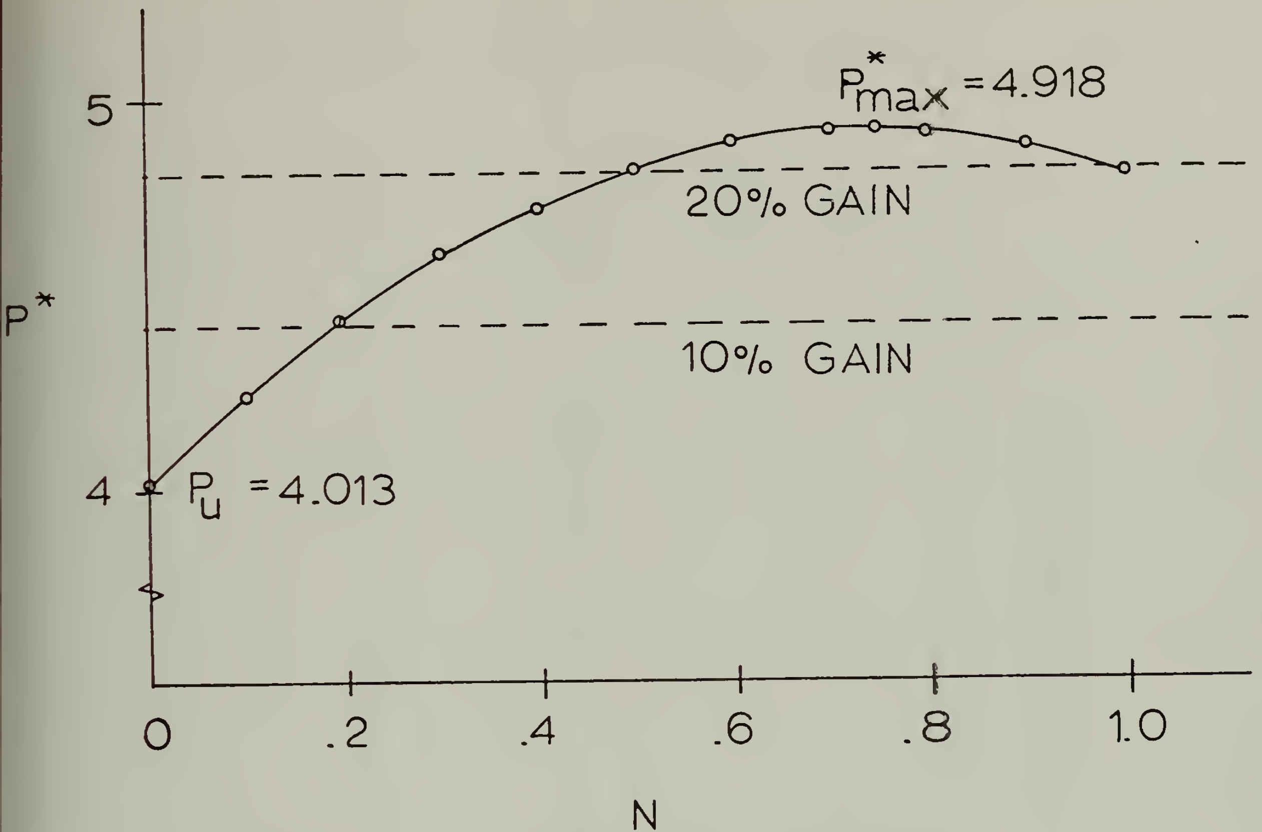

Figure 12. Amended Load P* Versus Values of N for Cantilever Beam.

The value n=0.14 provided a best buckling load of P* = 17.199 for the simply supported beam. This corresponds to a gain of only 1.55% over that of a constant rectangular cross section. For the cantilever, it was found that P* = 4.918 (n=0.75) which is a 22.5% gain over that of the rectangular section. Physically these results can be explained when the lateral bending and twisting moments of the two problems are examined.

Recalling equations (3.4) and (3.6) for the rectangular beam yields the following principle axes moments

$$M_\xi = -\frac{P}{2}(u_1 - u) + \frac{P}{2}\left(\frac{\ell}{2} - x\right)u' \qquad (4.41)$$

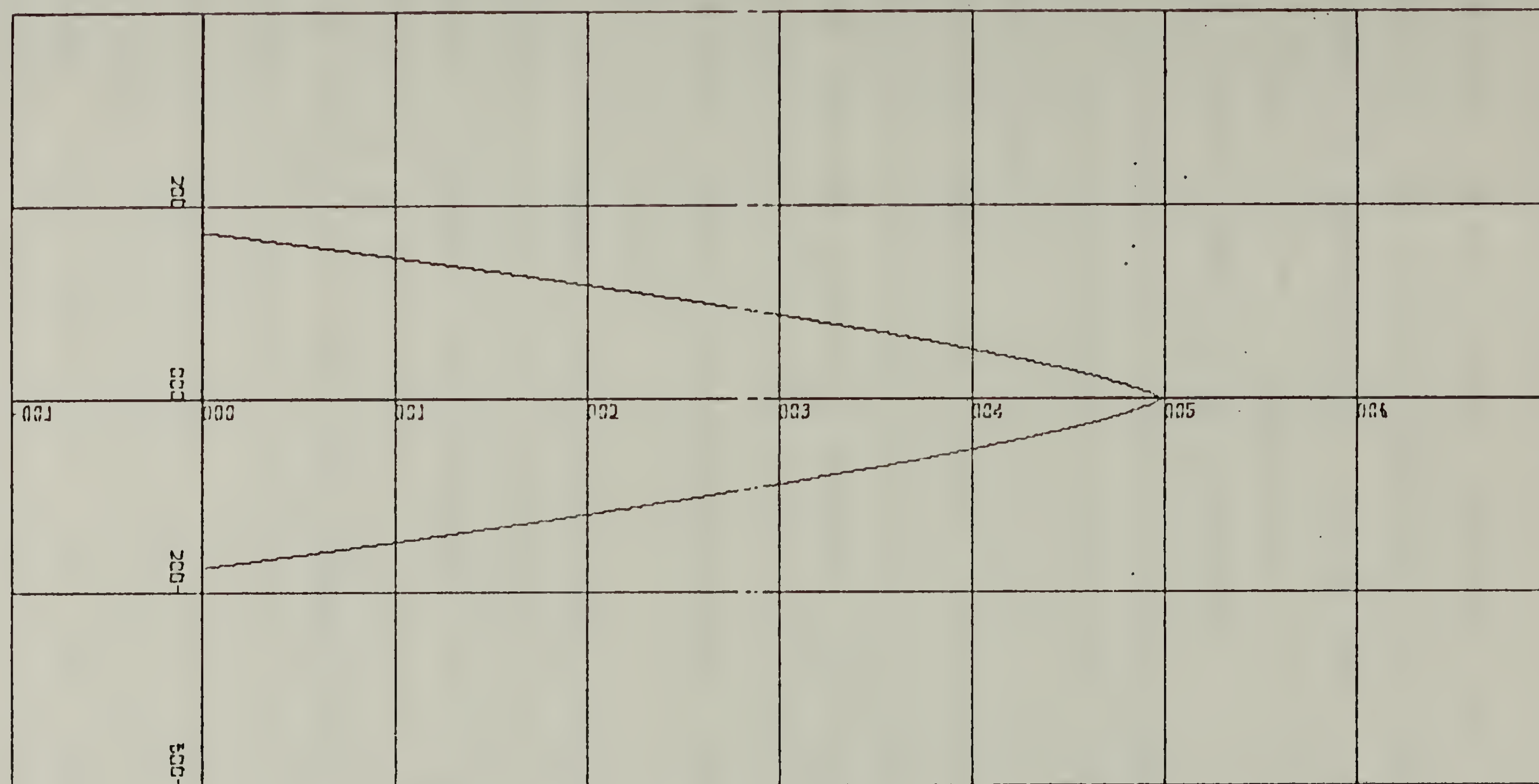

Figure 13. Cantilever Beam Design Case - $h = h_o(1 - x/L)^{0.75}$ $P^* = 4.918$.

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$$M_\eta = \frac{P}{2}\left(\frac{\ell}{2} - x\right)\phi \qquad (4.42)$$

Evaluation at the boundary condition, $x=0$: $u_1=u$ and $u'=0$ shows that the twisting moment $M_\xi=0$ and bending moment M_η assumes its maximum value. At the ends of the structure, $x = \frac{\ell}{2}$; $\phi=0$ and $u=0$. This implies that bending moment $M_\eta=0$ and twisting moment M_ξ is a maximum.

The combination of these two moments appear to restrict the beam from assuming an optimum shape which differs significantly from the uniform case. In fact, if the structures depicted in Appendix E are examined, it may be noted that as the beam approaches a pyramidal shape, the less stable the structure becomes. Physically, this seems to imply that the vanishing end area offers less and less resistance against the twisting moment, and the structure will readily buckle.

It is apparent and expected that the existing moment conditions govern the optimum shape of the structure. Physical intuition would indicate that the maximum amount of material be located where the maximum moment existed. If M_η and M_ξ were equal and a linear variation could be assumed, the resulting optimum shape would actually be that of a rectangular beam. For the problem considered, M_η appears to have a domineering influence on the optimum design. If this was not the case, then the solutions determined in Section IV-C would have indicated a "best" buckling load greater than those for the present solution.

The actual moment values for M_η and M_ξ are not known and cannot be determined unless the displacement and twist functions u and ϕ are known exactly. Additionally, these functions are coupled to the variable cross sectional area as indicated by equations (3.3 a,b,c). Analyzing equations (4.41) and (4.42) and Figure 4 leads to the following interpretation. The moment representation of M_ξ appears to be parabolic while that of M_η (based on a parabolic Φ) appears to be cubic and exhibits an inflection point. These patterns are shown in Figure 14. The maximum combined moment must then occur between x=0 and x= $\frac{\ell}{2}$.

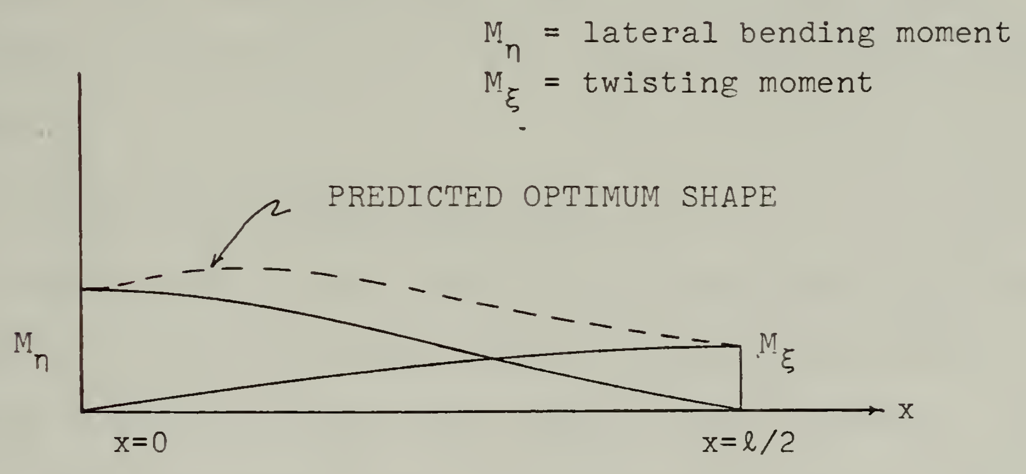


Figure 14. Expected Moment Configuration and Approximate Optimum Design.

On the basis of moment evaluation alone, the optimum design would apparently be of a form similar to the shape shown in Figure 14. Since M_η is expected to be greater than M_ξ the design would lump the largest amount of material somewhere between x=0 and x= $\frac{\ell}{2}$ but closest to x=0.

Available time has prevented analysis of this expected "optimum" shape. A Rayleigh Ritz formulation for this design was attempted by assuming the displacement and twist functions and using a Lagrange interpolating polynomial to approximate the design. The results indicated a buckling load slightly less than that of the uniform rectangular case.

Because of the low gain in efficiency (1.55%) from the best design obtained from the variable height solution, it appears that the true optimum shape will not reflect a large performance gain over that of the rectangular case. In fact, the determined value from the variable height solution is expected to be very close to the actual optimum design.

F. VALUES OF N GREATER THAN ONE

The analyses of the problems discussed have been restricted to the range for n=1. If n is greater than one the actual curvature of the beam design becomes concave. For the simply supported beam this physically implies that the structure must become less stable than the limiting pyramidal shape. This is evident since even less area is distributed toward the beam ends. But, it was found earlier that the pyramidal shape is actually unstable for any load. By this reasoning the problem must be restricted to the range of values used to have physical meaning.

Mathematically, if the value of n is allowed to be greater than one, then equation (4.15) changes to the form

$$\phi = \gamma^{\frac{1-n}{2}} [\alpha_1 J_{-\nu}(\rho) + \alpha_2 J_\nu(\rho)] \qquad (4.45)$$

for which α_1 must be zero. The resulting roots introduce a singularity condition and the solutions are misrepresentative of the problem. Consequently, it is deemed necessary for the value of n to be restricted to the interval $0 \leq n \leq 1$ to provide meaningful results.

V. CONCLUSIONS

For the simply supported beam the assumed power law restricts the number of designs which may be analyzed. However, a good indication of the problem behavior can be determined from the results. The number of designs which remain relatively stable do not differ radically from the uniform case. As the designs converge to a pyramidal shape they rapidly become less stable.

It appears that the combination of the two moments, M_η and M_ξ govern the design. As the beam approaches a pyramidal shape, less area becomes available to resist the twisting moment and the beam readily buckles. Thus the interaction of the two moments restrict the beam from attaining a shape which differs significantly from the rectangular case.

The effect of the moment values on beam design is better demonstrated from the cantilever results. At the free end of the beam there is no moment and the design calls for very little material to be located at that position. The interaction of the two moments seem to produce a maximum slightly away from the fixed end - otherwise, the design for the best buckling load would have indicated the pyramidal shape. The cantilever solution, thus, helps to support the validity of the simply supported beam results.

The true optimum shape for the simply supported beam has not been determined by the variable height formulation. However, the expected optimum shape was approximated from analyzing the physics of the problem. This shape is not expected to offer any significant increase over the design obtained from the variable height formulation.

APPENDIX A

A STRUCTURAL OPTIMIZATION PROBLEM

An example of a structural optimization problem which illustrates the usefulness of the variational method of Section II is that of obtaining the optimum shape of a uniformly loaded beam. This problem is a "strength" problem in contrast to the buckling problem which exhibits no strength. For the problem considered, bending is in the vertical plane of symmetry (Fig. 15). The height of the

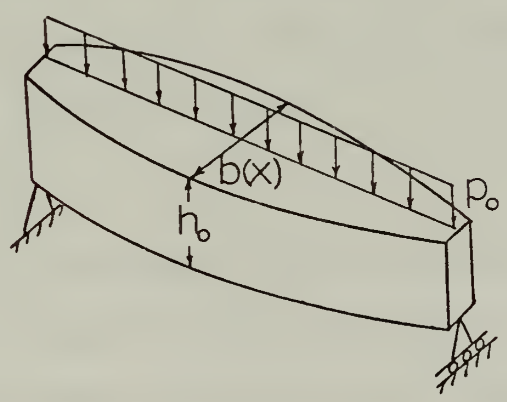


Figure 15. Variable Width Uniformly Loaded Beam.

beam is taken as constant, and a uniform load is applied.

1. Statement of the Problem

The design b(x) is to be determined for a variable width which provides a maximum strength capacity for the structure. Unless some specific constraints are imposed on the problem, results may well be meaningless. For instance, if a minimum volume structure was considered with no other

conditions than equilibrium and boundary conditions to be satisfied, then it is entirely reasonable to deduce a zero volume structure with infinite strength capacity [6].

For the strength problem the primary constraint is that the material must not fail. Thus, for an elastic structure composed of material of finite strength capacity, the specific strain energy (ψ) must be no greater than a specified maximum. This maximum is denoted as $\bar{\psi}$, such that,

$$\psi \leq \bar{\psi} \qquad (A.1)$$

at any point in the structure. The inequality of (A.1) may be written as an equality constraint by introducing a scaler variable $\gamma(\bar{x})$ to take up any existing slack in the function. Using the "slack variable" gives

$$\psi + \gamma^2 = \bar{\psi}\,. \qquad (A.2)$$

For the simply supported beam the moment of inertia with respect to the principle bending axis is $I = \frac{1}{12}\,b(x)h_0^3$ or $I = CA(x)$. The strain energy per unit length at any position along the beam is known to be $1/2ECAv''^2$ where v represents the displacement function. If the strain energy per unit length is divided by the cross sectional area, A, at any position, then specific strain energy per unit area per unit length may be written

$$\psi = 1/2ECv''^2. \qquad (A.3)$$

An energy constraint equation for the problem is then determined,

$$\bar{\psi} - (\frac{1}{2} ECv''^2 + \gamma^2) = 0 . \qquad (A.4)$$

A second constraint on the problem restricts the beam to be designed from a given volume of material V_o. This is the isoperimetric constraint introduced in Section II. Thus

$$\int_0^{\ell} A(x)dx = V_o . \qquad (A.5)$$

The total potential energy of the system may be written

$$T = \int [\frac{1}{2} ECAv''^2 - P_o v] \, dx, \qquad (A.6)$$

where the first term represents the strain energy due to bending and the second term defines the external work of the system due to p_o[10].

The volume constraint (A.5) and energy constraint (A.4) are now accommodated through the formation of the augmented functional T*,

$$T^* = \int_0^{\ell} [\frac{1}{2}ECAv''^2 - p_o v - \lambda_1 A - \lambda_2(\bar{\psi} - \frac{1}{2}ECv''^2 + \gamma^2)]dx \qquad (A.7)$$

where λ_1 and λ_2 are Lagrangian multipliers.

Performing a variation with respect to v, A, and γ yields three Euler equations.

$$\delta_v T^*=0: \quad [ECAv'' + ECv''\lambda_2]'' - p_o = 0 \qquad (A.8)$$

$$\delta_A T^*=0: \quad \tfrac{1}{2}ECv''^2 - \lambda_1 = 0 \qquad (A.9)$$

$$\delta_\gamma T^*=0: \quad 2\lambda_2 \gamma = 0 . \qquad (A.10)$$

The natural boundary conditions are v(0)=v(L)=0 and M(0)=M(L)=0.

Integrating equation (A.8) twice yields

$$EC(Av'' + \lambda_2 v'') = \tfrac{1}{2}\, p_o X^2 + C_1 X + C_2 \ . \qquad (A.11)$$

for which the above boundary conditions determine $C_1 = \frac{-p_o}{2}$ and $C_2 = 0$. Consequently (A.11) becomes

$$v''(A+\lambda_2) = \frac{p_o}{2EC}\,(X^2 - \ell x). \qquad (A.12)$$

Using equation (A.9) gives

$$v'' = \left(\frac{2\lambda_1}{EC}\right)^{\frac{1}{2}} \qquad (A.13)$$

which physically implies that the deflection curvature is constant.

From equation (A.10) there exist three possibilities

$$\text{i)} \quad \lambda_2 \equiv 0, \gamma \equiv 0 \qquad (A.14)$$

$$\text{ii)} \quad \lambda_2 \not\equiv 0, \gamma \equiv 0 \qquad (A.15)$$

$$\text{iii)} \quad \lambda_2 \equiv 0, \gamma \not\equiv 0\ . \qquad (A.16)$$

Assuming $\gamma = 0$ and examining equation (A.4), it may be observed that

$$v'' = \left(\frac{2\bar{\psi}}{EC}\right)^{\frac{1}{2}}. \qquad (A.17)$$

Comparing this with (A.13) implies

$$\bar{\psi} = \lambda_1\ . \qquad (A.18)$$

Recalling [11] for a simply supported beam under a uniform load Po, the moment at any position X along the beam may be represented by

$$M(x) = \frac{p_o}{2}\,(X^2 - \ell x) \qquad (A.19)$$

then equation (A.12) may be rewritten

$$v''(A+\lambda_2) = \frac{1}{EC} M(x) . \tag{A.20}$$

But, the moment is also known as M=ECAv"; consequently,

$$v''(A+\lambda_2) = v''A . \tag{A.21}$$

From the above equation it is obvious that $\lambda_2 \equiv 0$ and (A.14) is satisfied. This condition implies that the energy formulation actually rejects an energy constraint. This can be observed by examining equation (A.8), which should be the equilibrium equation of the system. The condition is valid only if λ_2 is, in fact, zero. The rejection of an energy constraint appears to be inherent to the energy formulation.

This formulation seeks to maximize the maximum stress at every position x (i.e. strain energy density) which is different than maximizing specific strain energy as was formulated in the problem.

Therefore, equation (A.21) becomes

$$v''A(x) = \frac{p_o}{2EC} (x^2 - \ell x) \tag{A.22}$$

from which the variable area may be determined

$$A(x) = \frac{p_o(x^2 - \ell x)}{(8 \psi EC)^{\frac{1}{2}}} . \tag{A.23}$$

Substituting for A(x) into the isoperimetric equation (A.5) and integrating, the load Po is found

$$p_o = - \frac{6V_o}{\ell^3} (8\bar{\psi}EC)^{\frac{1}{2}} . \tag{A.24}$$

The result determines the optimum shape of the cross sectional area, or in terms of the width

$$b(x) = \frac{6V_o}{\ell^3 h_o} (\ell x - x^2) . \tag{A.25}$$

Since the stress at any section may be represented as $\sigma = \frac{Mc}{I}$, substitution for M and c ($c = h_o/2$) yields

$$\sigma = \frac{Eh_o}{2} \left(\frac{2\bar{\Psi}}{EC}\right)^{\frac{1}{2}} = \frac{Eh_o}{2} \left(\frac{2\lambda}{EC}\right)^{\frac{1}{2}} = \bar{\sigma} . \tag{A.26}$$

This represents the maximum allowable stress at any section along the beam. It is noted additionally, that the stress is independent of any position x along the beam. Thus, the stress is maximized at the outer fibers ($h_o/2$) for any position with respect to the optimum shape. The optimum shape is noted to be parabolic and is shown below (Fig.16) with stresses indicated. The design is comparable to the uniform strength design listed by Esbach [20].

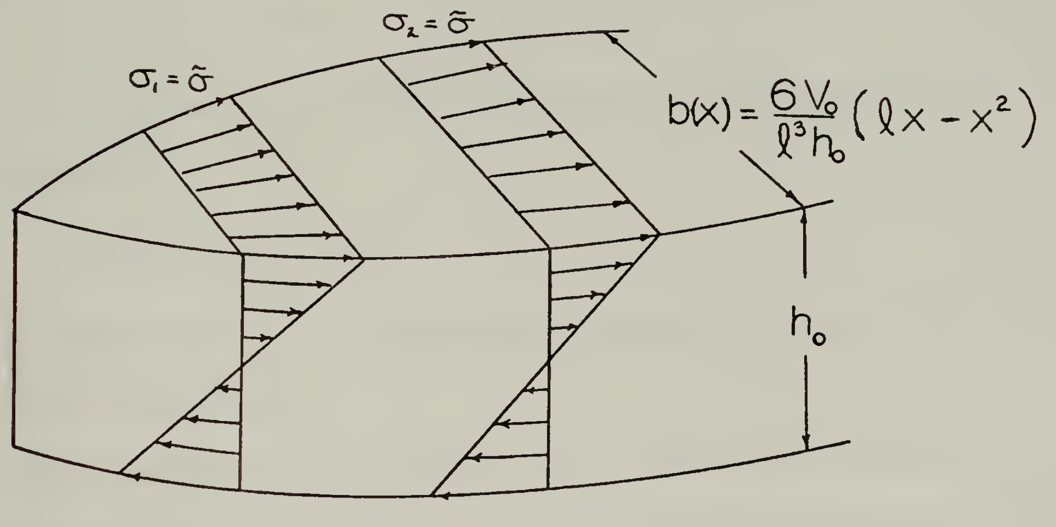


Figure 16. Optimum Shape for Variable Width Beam Design.

APPENDIX B

EFFECT OF THE LOAD P AT A DISTANCE FROM THE CENTROID

In Section III the assumption was made that the load P should be located at the centroid of the beam. The effect of the geometrical location of the load is significant in the energy formulation analysis. For the lateral buckling case, the external work associated with the unstable state is represented by

$$P \int_0^{\ell/2} \phi u'' (\ell/2 - x)\, dx \qquad (3.20)$$

This condition results from analyzing a longitudinal element dx at point D in Figure 4. Considering bending of this element in the $\xi\zeta$ plane with the cross section mn assumed fixed, the end of the beam then describes an infinitely small arc

$$u''(\ell/2 - x)\, dx \qquad (B.1)$$

in the $\xi\zeta$ plane, for which the vertical component may be represented

$$\phi u''(\ell/2 - x)\, dx \qquad (B.2)$$

The external work is P times the integral of this term over the area considered.

Now consider the case where the load P is located at the top of the beam (Figure 17). The external work must now be determined from lowering of the load by amounts Δ_1 and Δ_2 [19].

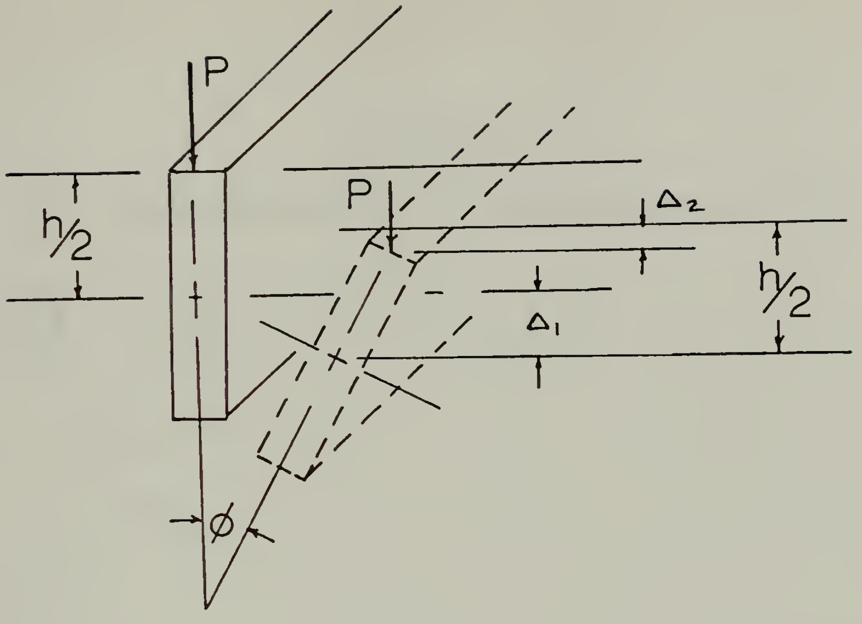


Figure 17. Effect of the Load P Located at Top of Beam.

Δ_1 is simply lowering of the load due to the change in centroidal height during lateral buckling and is the same as (3.20). The component Δ_2 represents the lowering due to twisting of the beam through an angle ϕ and can be represented as $h/2\ (1 - \text{Cos}\ \phi) \simeq \frac{a\phi^2}{2}$. Thus an additional term must be considered if the load is placed away from the beam centroid. This term reduces to $\frac{Ph}{4}\ \phi^2$.

APPENDIX C

TRANSFORMATION OF EQUATION (4.11)

$$\frac{d^2\phi}{d\gamma^2} + \frac{n}{\gamma}\frac{d\phi}{d\gamma} + K^2\gamma^{2(1-n)}\phi = 0 \qquad (4.11)$$

$$\rho = \frac{K}{2-n}\gamma^{2-n} \qquad (4.12a)$$

$$\phi = \gamma^{\frac{1-n}{2}}\beta(\gamma) \qquad (4.12b)$$

from (4.12b)

$$\frac{d\phi}{d\gamma} = \frac{1-n}{2}\gamma^{\frac{-1-n}{2}}\beta + \gamma^{\frac{1-n}{2}}\frac{d\beta}{d\gamma} \qquad (C.1)$$

where

$$\frac{d\beta}{d\gamma} = \frac{d\beta}{d\rho}\frac{d\rho}{d\gamma} \qquad (C.2)$$

Substituting to get

$$\frac{d\phi}{d\gamma} = \frac{1-n}{2}\gamma^{\frac{-1-n}{2}}\beta + \gamma^{\frac{3}{2}(1-n)}K\frac{d\beta}{d\rho} \qquad (C.3)$$

Differentiating again

$$\frac{d^2\phi}{d\gamma^2} = \frac{(1-n)}{2}\frac{(-1-n)}{2}\gamma^{\frac{-3-n}{2}}\beta + 2(1-n)K\gamma^{\frac{-1-3n}{2}}\frac{d\beta}{d\rho}$$

$$+ K^2\gamma^{5/2\,(1-n)}\frac{d^2\beta}{d\rho^2} \qquad (C.4)$$

Substituting from (4.12b), (C.3) and (C.4) into (4.11) and arranging in tabular form:

$\frac{d^2\beta}{d\rho^2}$	$\frac{d\beta}{d\rho}$	β	
$K^2\gamma^{\frac{5}{2}(1-n)}$	$2(1-n)K\gamma^{\frac{1-3n}{2}}$	$\frac{n^2-1}{4}\gamma^{\frac{-3-n}{2}}$	$\frac{d^2\phi}{d\gamma^2}$
	$nK\gamma^{\frac{1-3n}{2}}$	$\frac{n(n-1)}{2}\ \gamma^{\frac{-3-n}{2}}$	$\frac{n}{\gamma}\ \frac{d\phi}{d\gamma}$
		$K^2\gamma^{\frac{5}{2}(1-n)}$	$K^2\gamma^{2(1-n)}$

$$K^2\gamma^{\frac{5}{2}(1-n)}\ \frac{d^2\beta}{d\rho^2} + (2-n)K\gamma^{\frac{(1-3n)}{2}}\ \frac{d\beta}{d\rho}$$

$$+ \left[\left(-\ \frac{n^2}{4} + \frac{n}{2} - \frac{1}{4}\right)\gamma^{\frac{-3-n}{2}} + K^2\gamma^{\frac{5}{2}(1-n)} \right]\beta \qquad (C.5)$$

The operator from equation (C.4) is shown at the right side of the table.

Substituting (4.12a) and letting

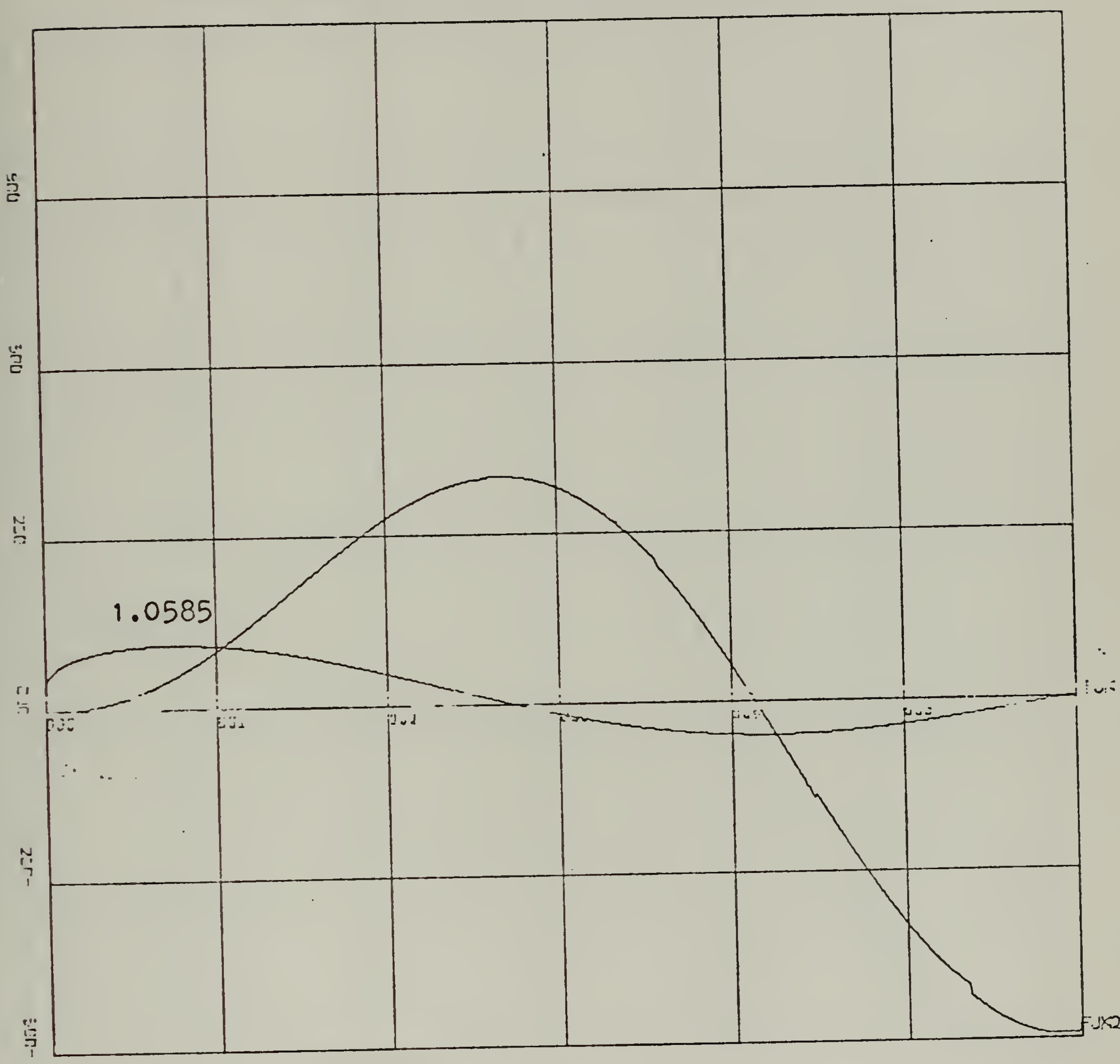
$$\nu = \frac{1-n}{2(2-n)} \qquad (C.6)$$

Equation (5) becomes

$$\frac{d^2\beta}{d\rho^2} + \frac{1}{\rho}\ \frac{d\beta}{d\rho} + \left(1 - \frac{\nu^2}{\rho^2}\right)\ \beta = 0 \qquad (4.13)$$

APPENDIX D

BESSEL EQUATION CURVES FOR LATERAL BUCKLING



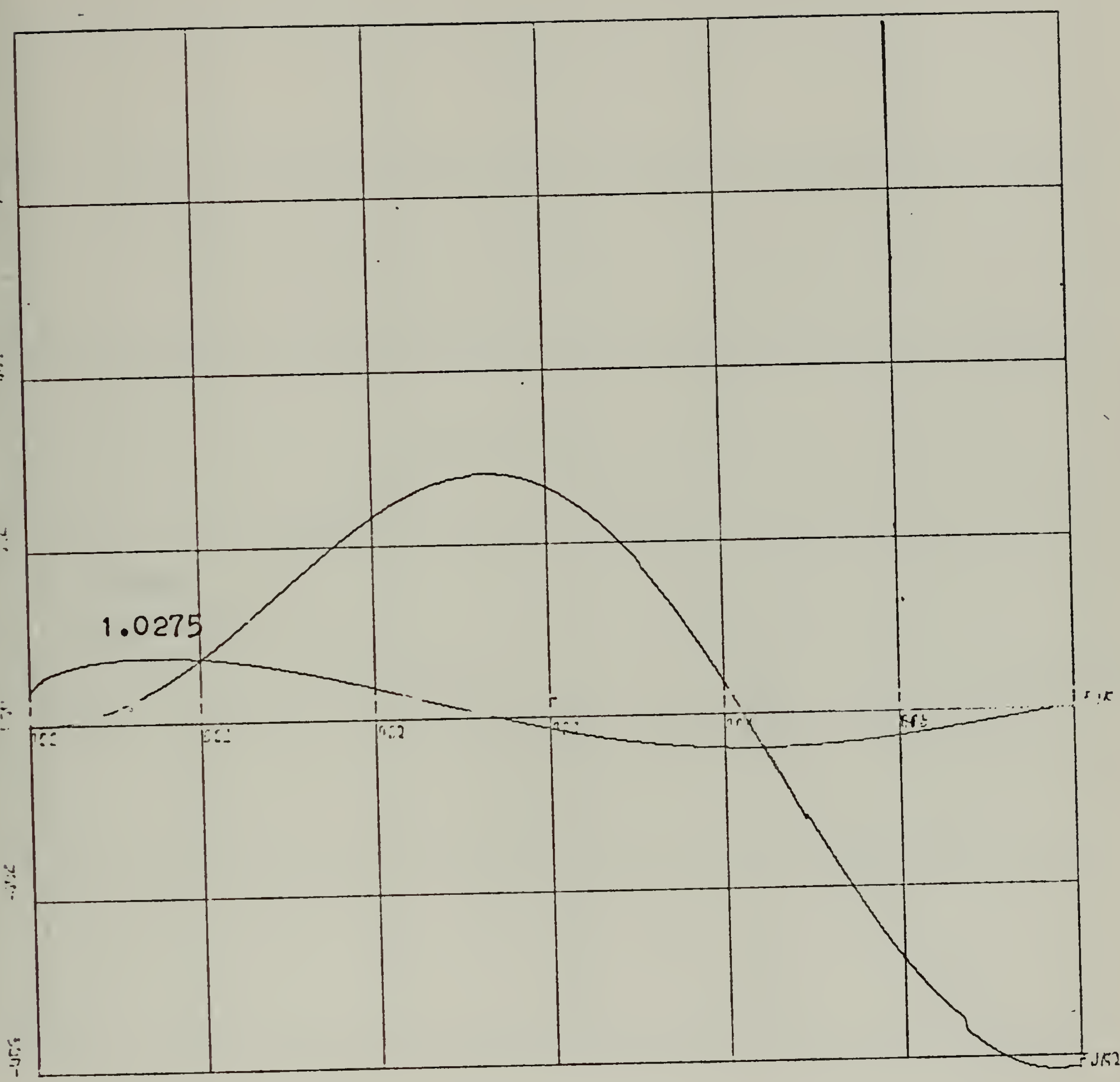
X-SCALE=1.00E+00 UNITS INCH. CASE: N=0

Y-SCALE=2.00E+00 UNITS INCH.

R. L. BURNS LCDR, USN

BESSEL EQ. CURVES FØR BUCKLING

Figure 18.



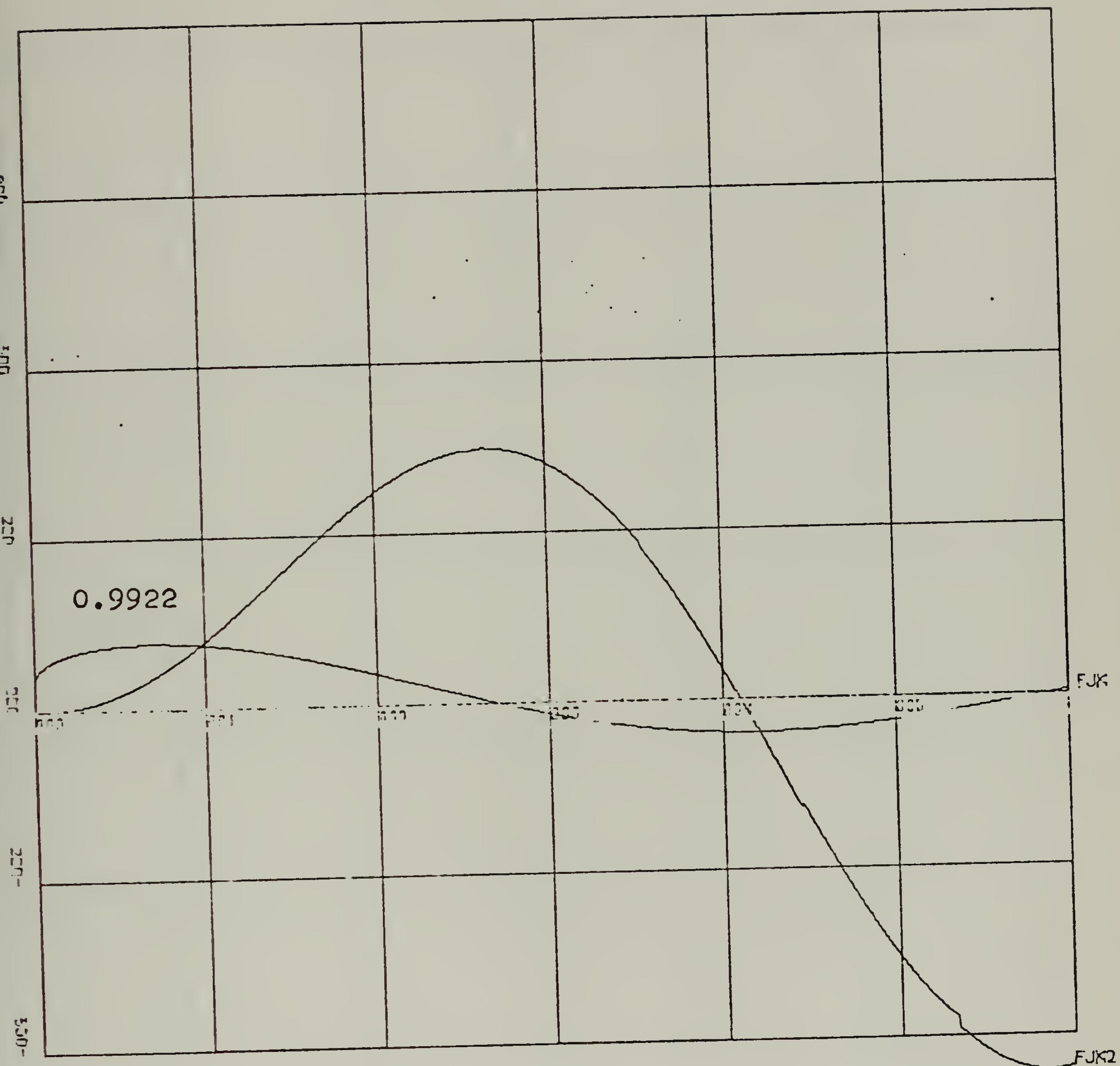
X SCALE=1.00E+00 UNITS INCH. CASE: N=0.1.

Y-SCALE=2.00E+00 UNITS INCH.

R. L. BURNS LCDR, USN

BESSEL EQ. CURVES FOR BUCKLING

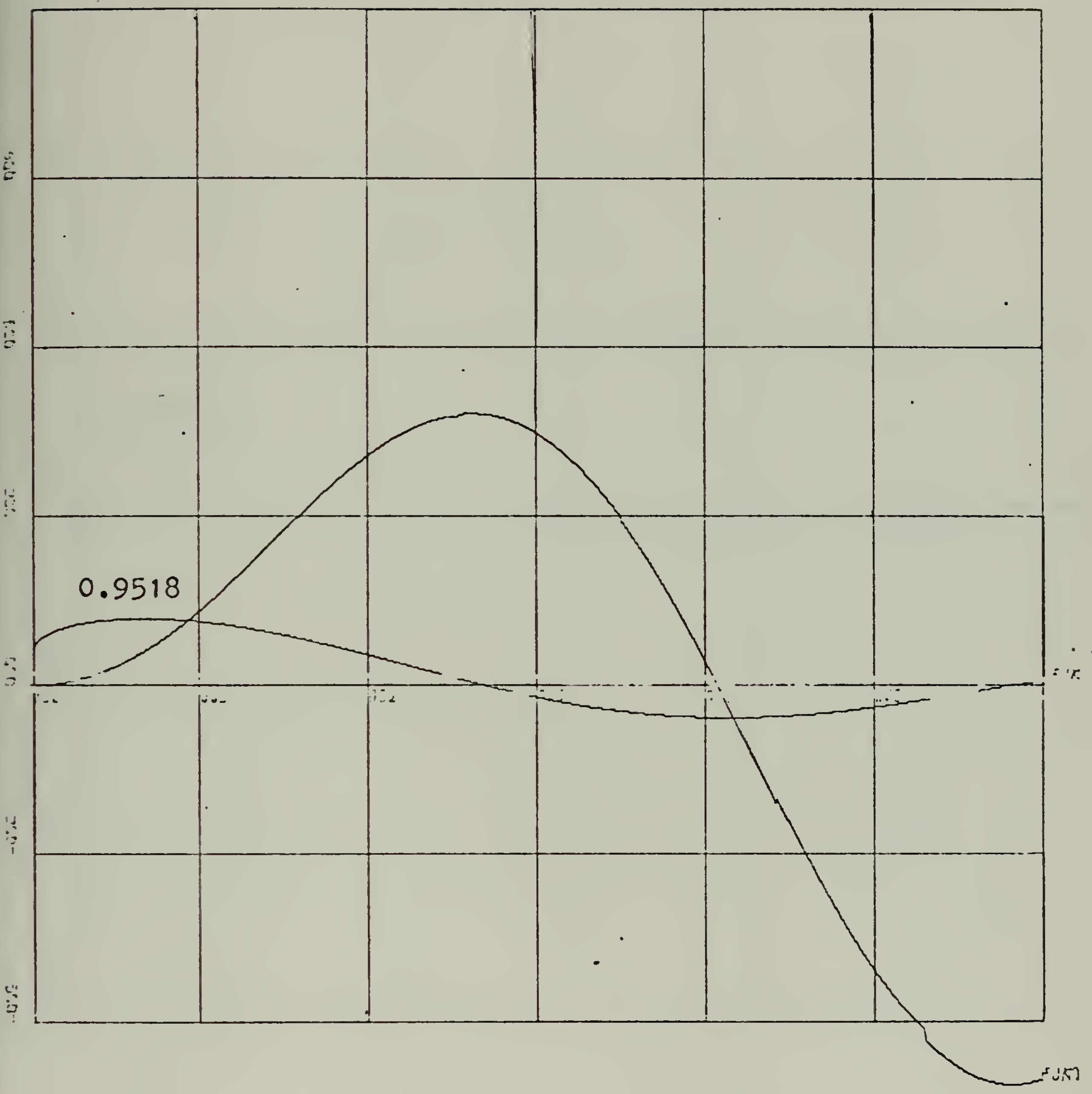
Figure 19.



X-SCALE=1.00E+00 UNITS INCH. CASE: N=0.2
Y-SCALE=2.00E+00 UNITS INCH.

R. L. BURNS LCDR, USN
BESSEL EQ. CURVES FØR BUCKLING

Figure 20.

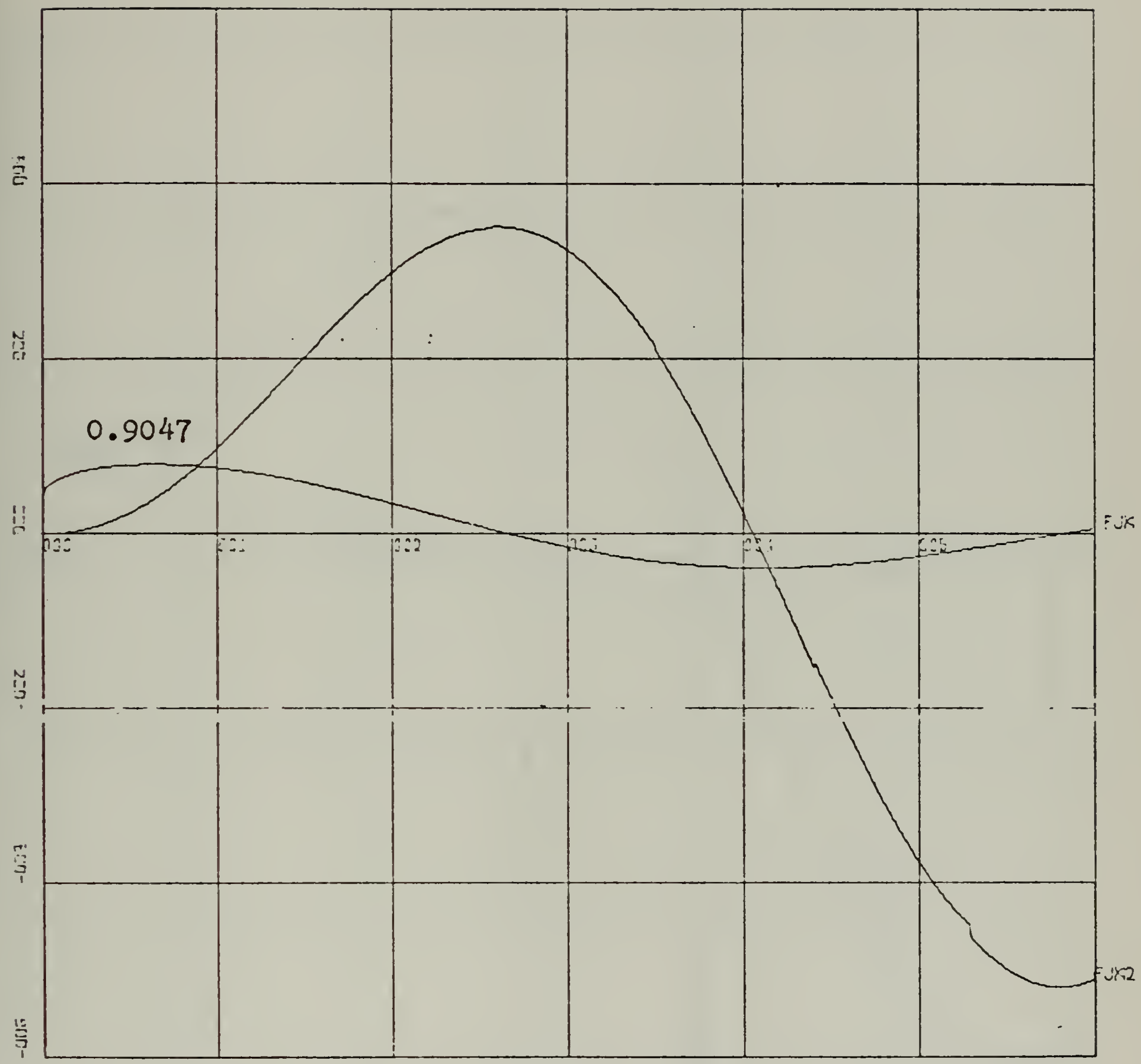


X-SCALE=1.00E+00 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

CASE: N=0.3

R. L. BURNS LCDR, USN
BESSEL EQ. CURVES FOR BUCKLING

Figure 21.



K-SCALE=1.00E+00 UNITS INCH.

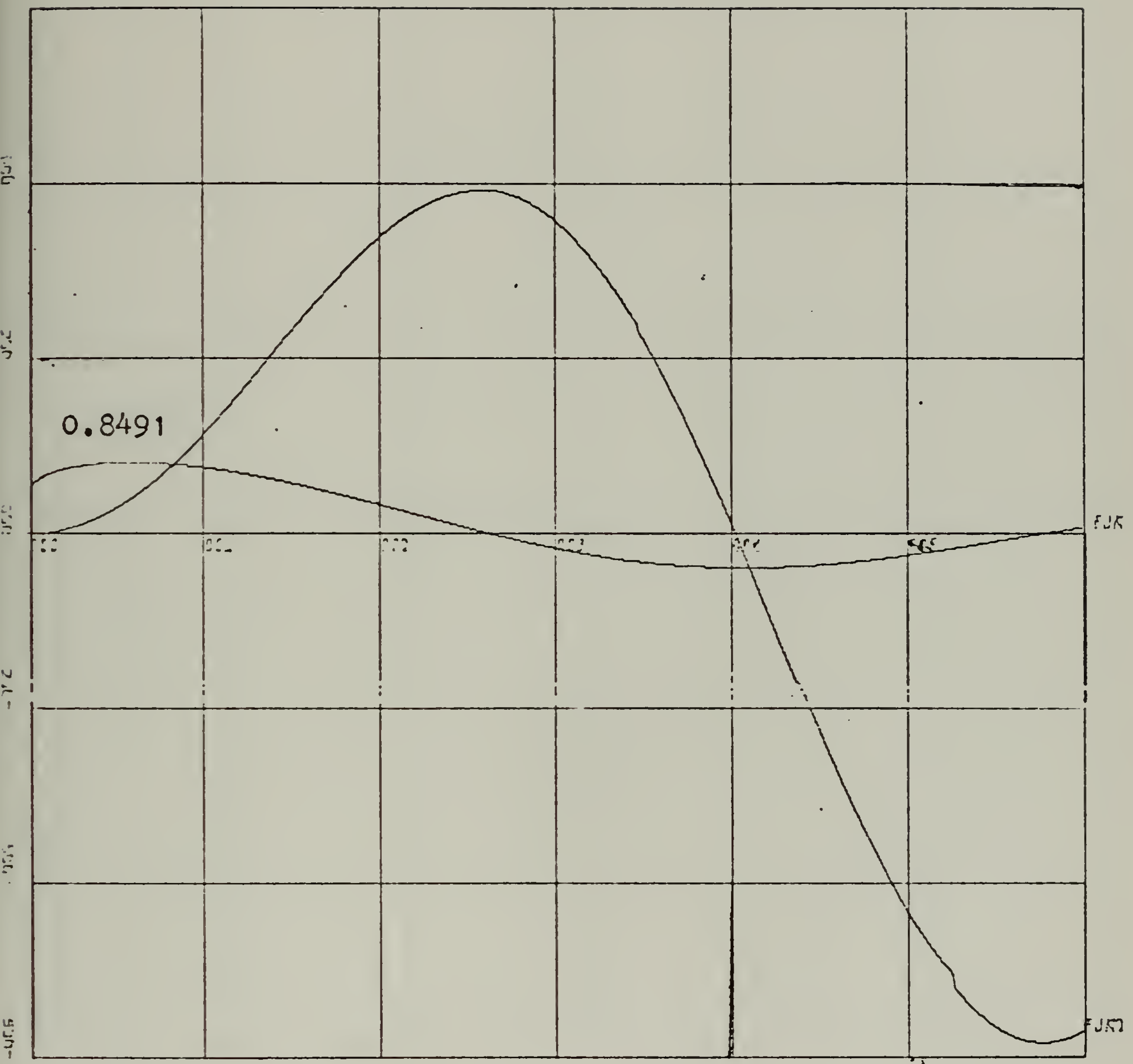
Y-SCALE=2.00E+00 UNITS INCH.

CASE: N=0.4

R. L. BURNS LCDR, USN

BESSEL EQ. CURVES FØR BUCKLING

Figure 22.

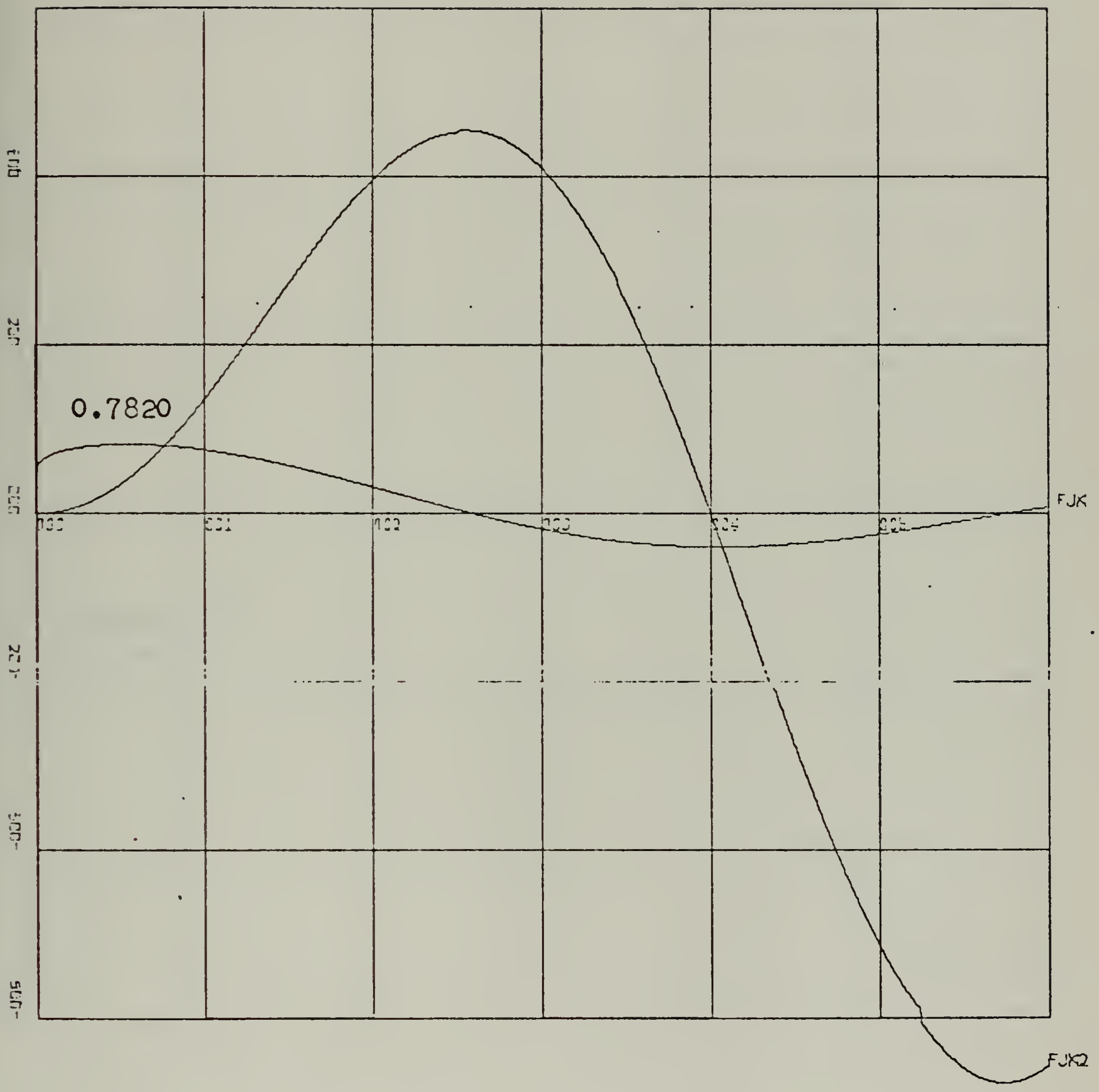


X-SCALE=1.00E+00 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

CASE: N=0.5

R. L. BURNS LCDR, USN
BESSEL EQ. CURVES FOR BUCKLING

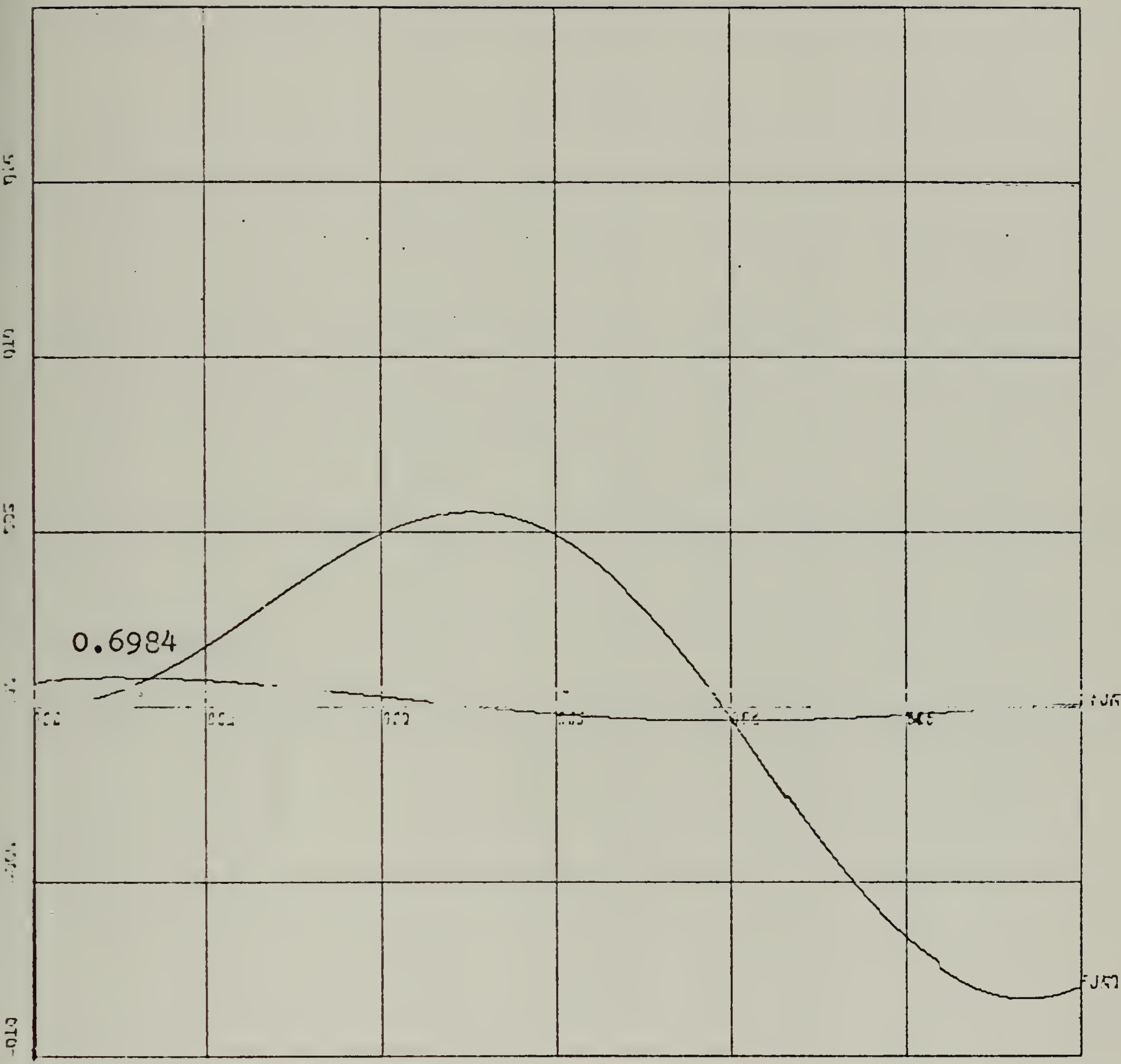
Figure 23.



X-SCALE=1.00E+00 UNITS INCH. CASE: N=0.6
Y-SCALE=2.00E+00 UNITS INCH.

R. L. BURNS LCDR, USN
BESSEL EQ. CURVES FØR BUCKLING

Figure 24.

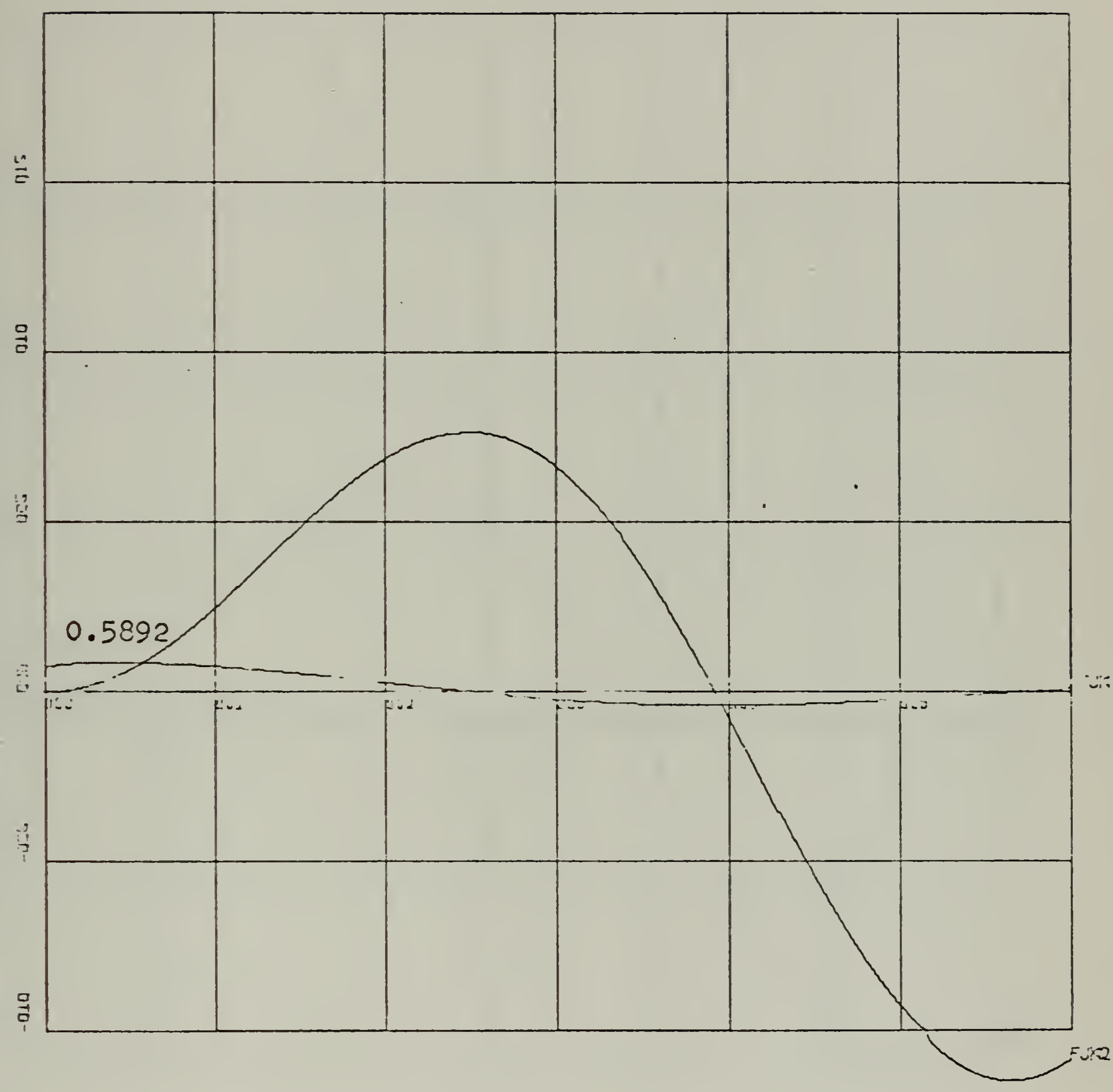


X-SCALE=1.00E+00 UNITS INCH.
Y-SCALE=5.00E+00 UNITS INCH.

CASE: N=0.7

R. L. BURNS LCDR, USN
BESSEL EQ. CURVES FOR BUCKLING

Figure 25.

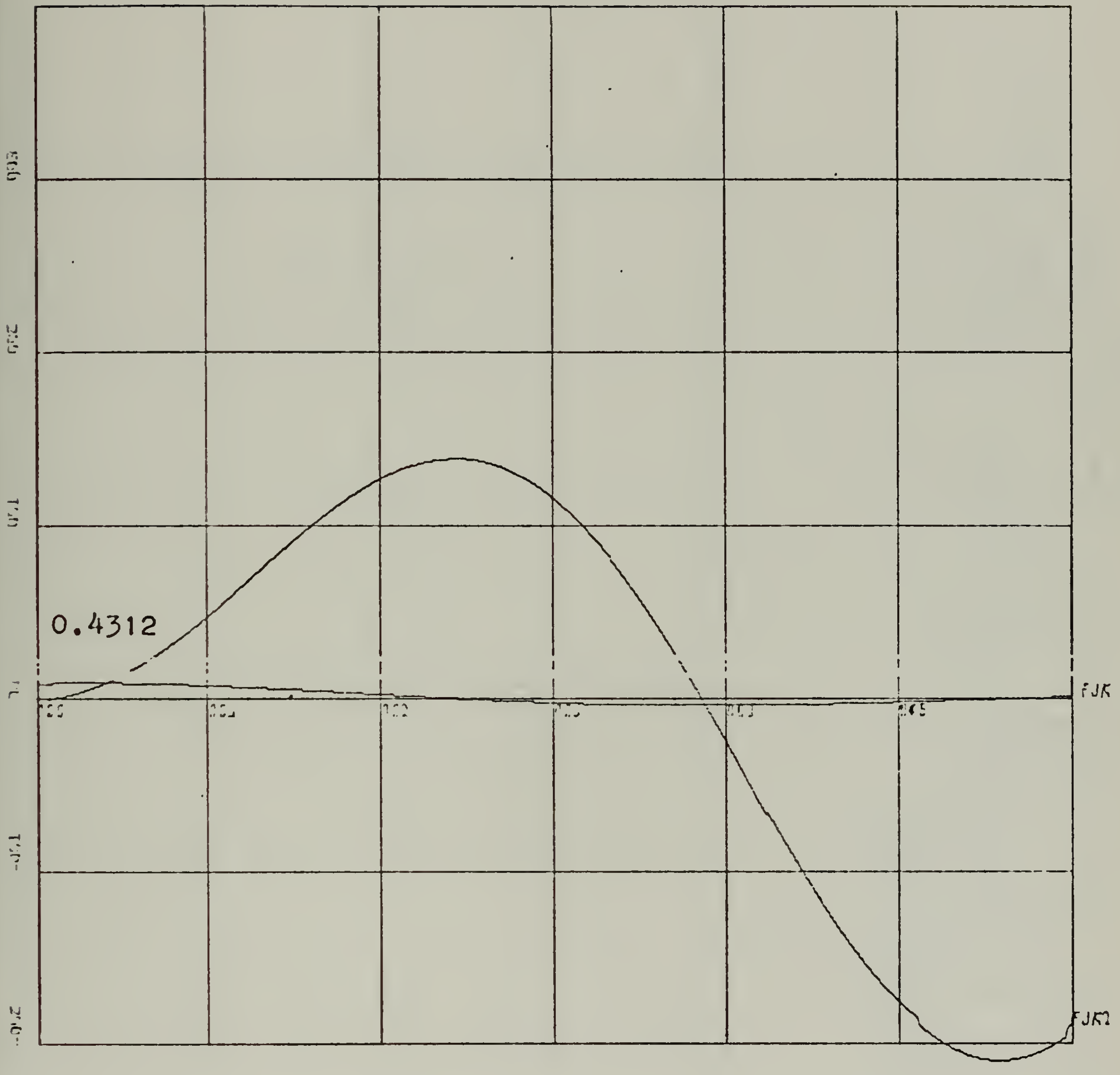


X-SCALE=1.00E+00 UNITS INCH.
Y-SCALE=5.00E+00 UNITS INCH.

CASE: N=0.8

R. L. BURNS LCDR, USN
BESSEL EQ. CURVES FØR BUCKLING

Figure 26.



X-SCALE=1.00E+00 UNITS INCH.
Y-SCALE=1.00E+01 UNITS INCH.

CASE: N=0.9

R. L. BURNS LCDR, USN
BESSEL EQ. CURVES FOR BUCKLING

Figure 27.

APPENDIX E

VARIABLE HEIGHT BEAM DESIGNS

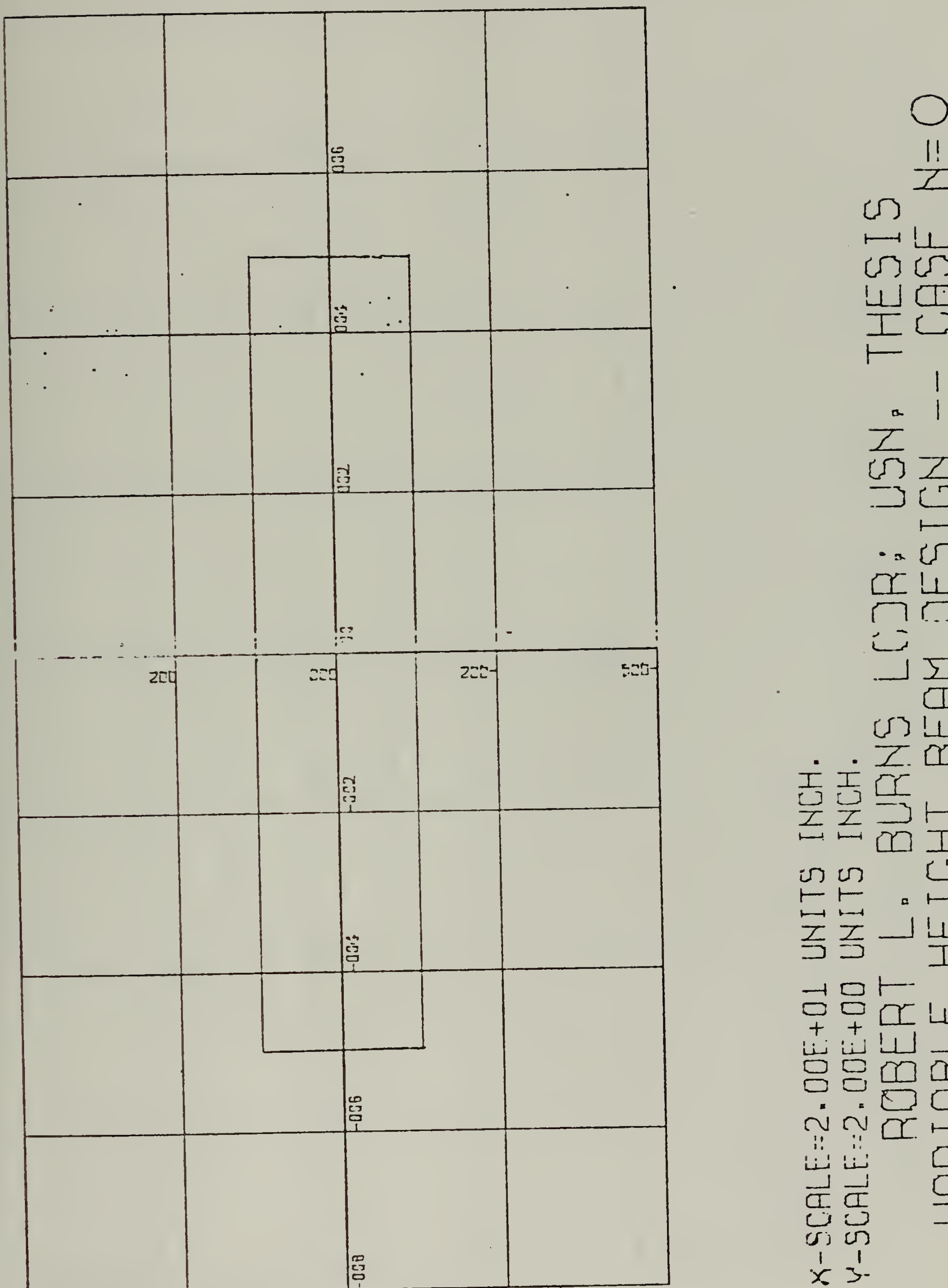
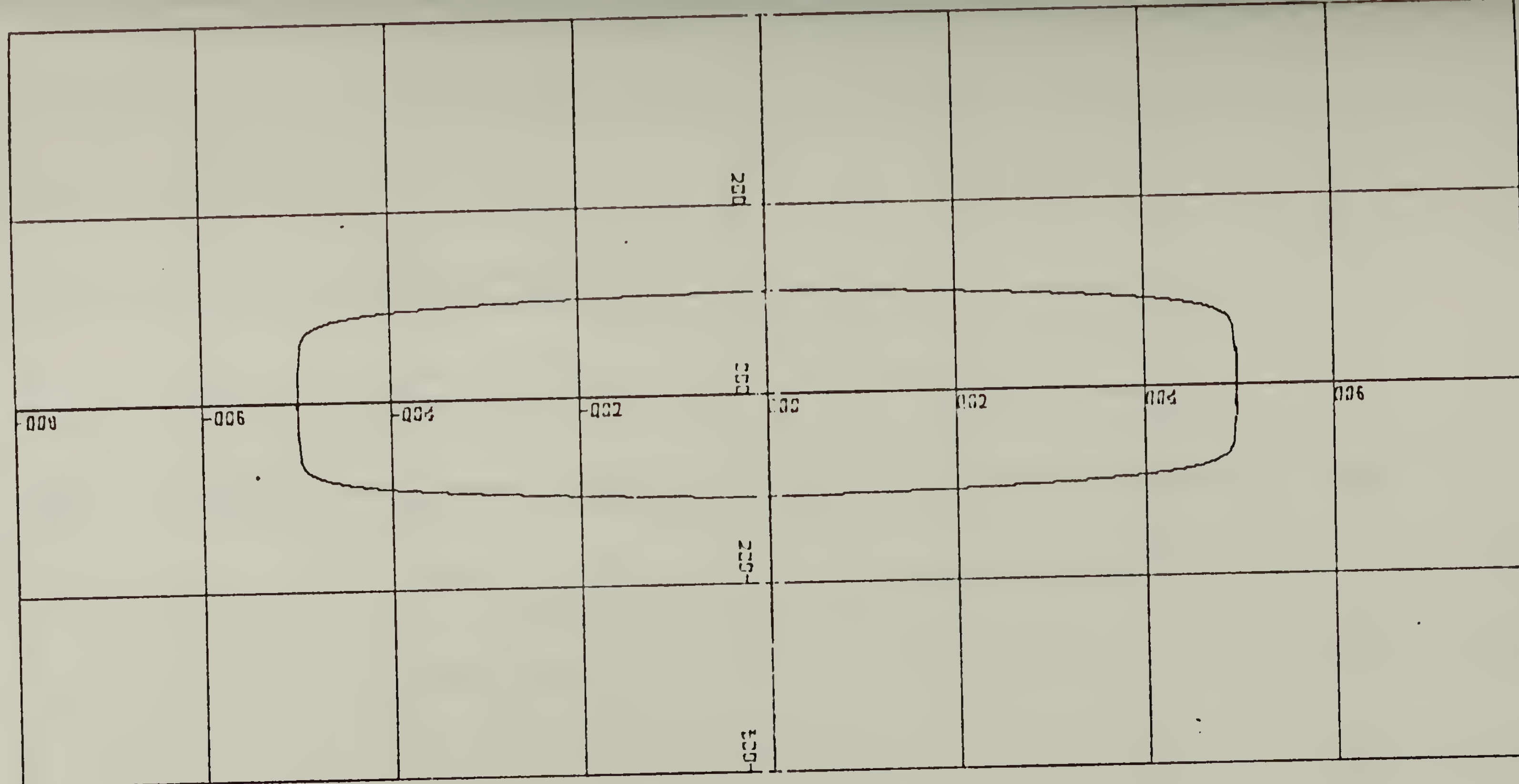


Figure 28. Simply Supported Beam - Case $h=h_o(1-2x/L)^0$

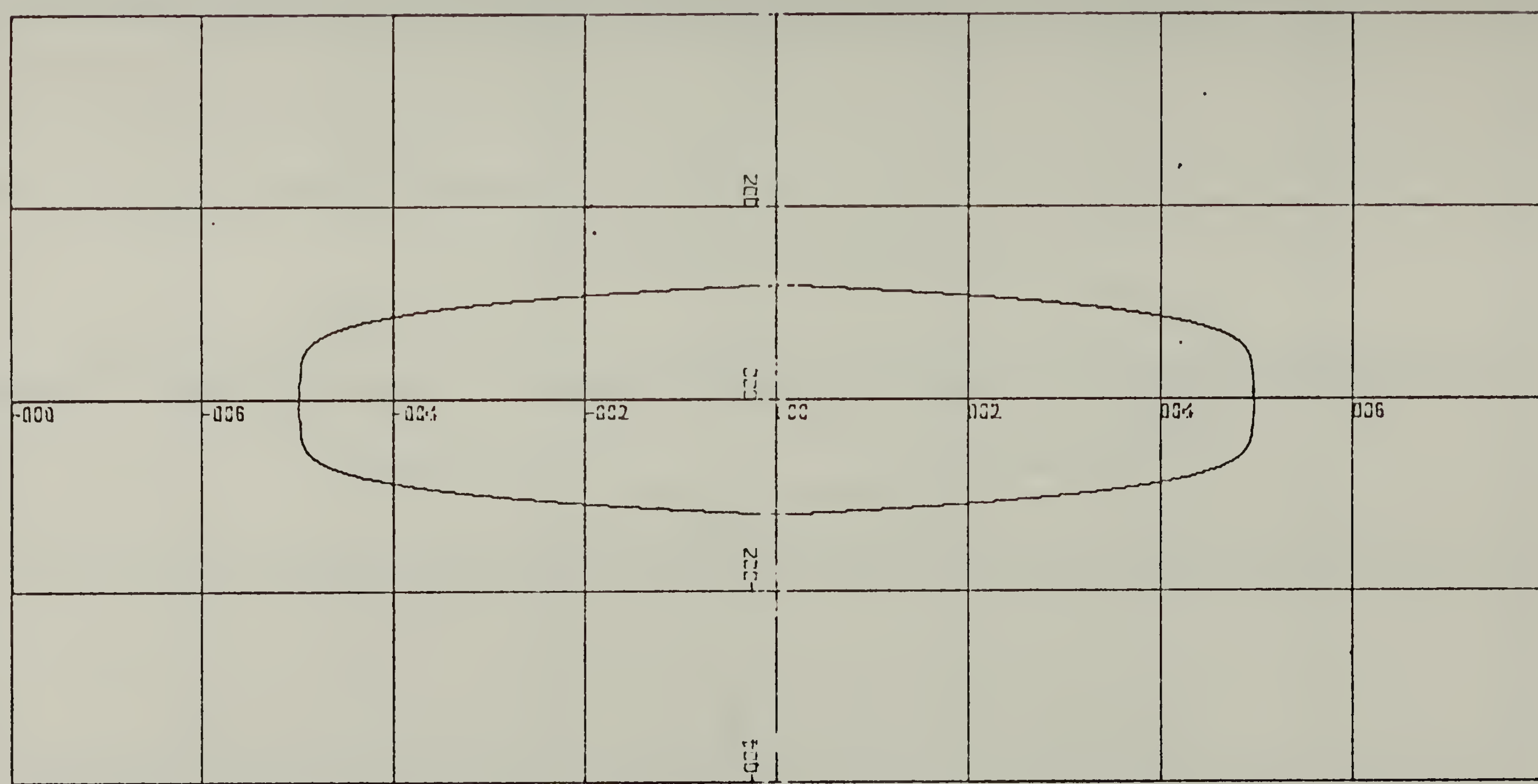
$P^* = 16.936$



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.1

Figure 29. Simply Supported Beam - Case $h = h_o(1-2x/L)^{0.1}$.
$P^* = 17.179$

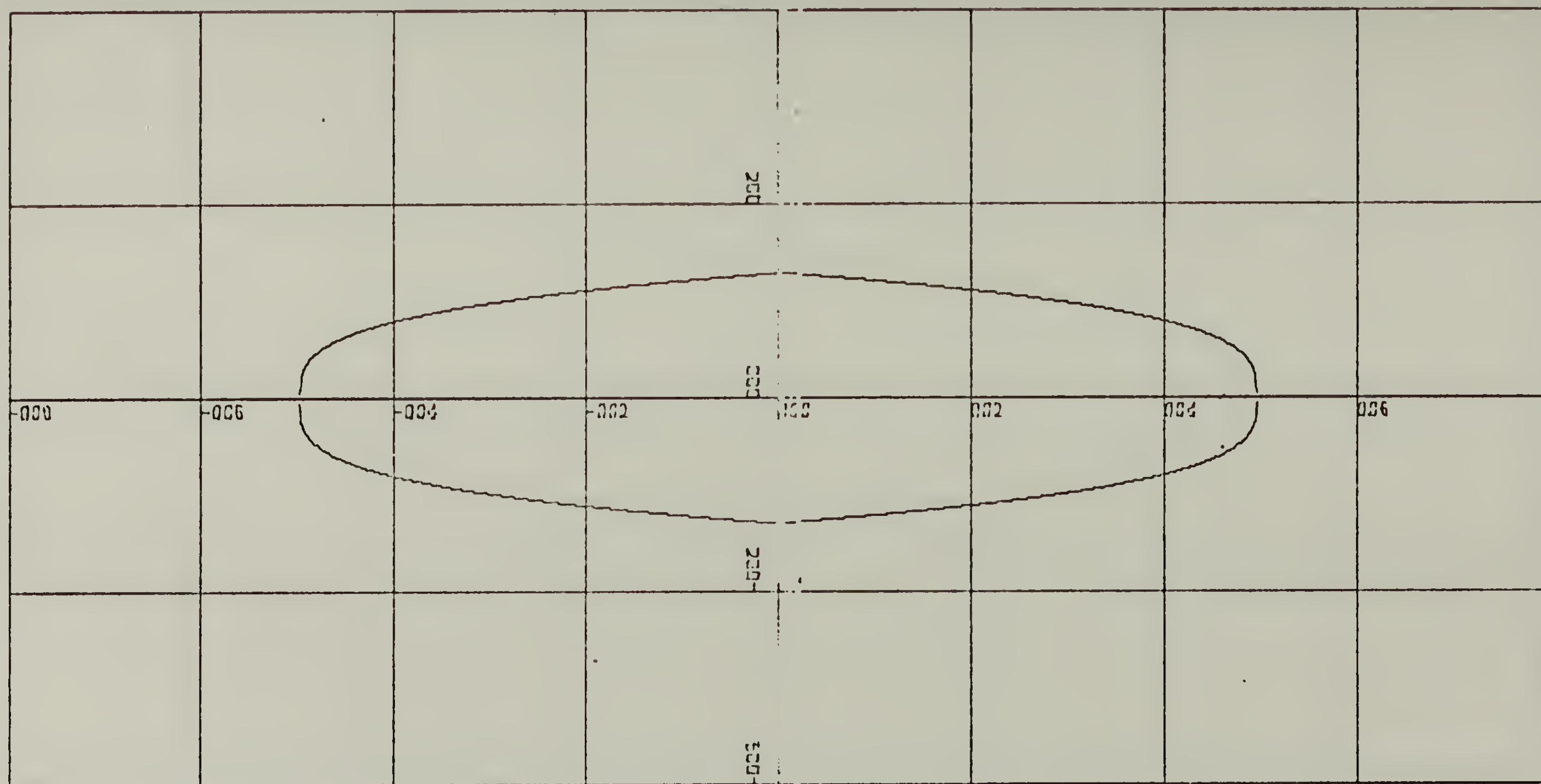


X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

RØBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.2

Figure 30. Simply Supported Beam - Case $h=h_o(1-2x/L)^{0.2}$
$P^*=17.145$.

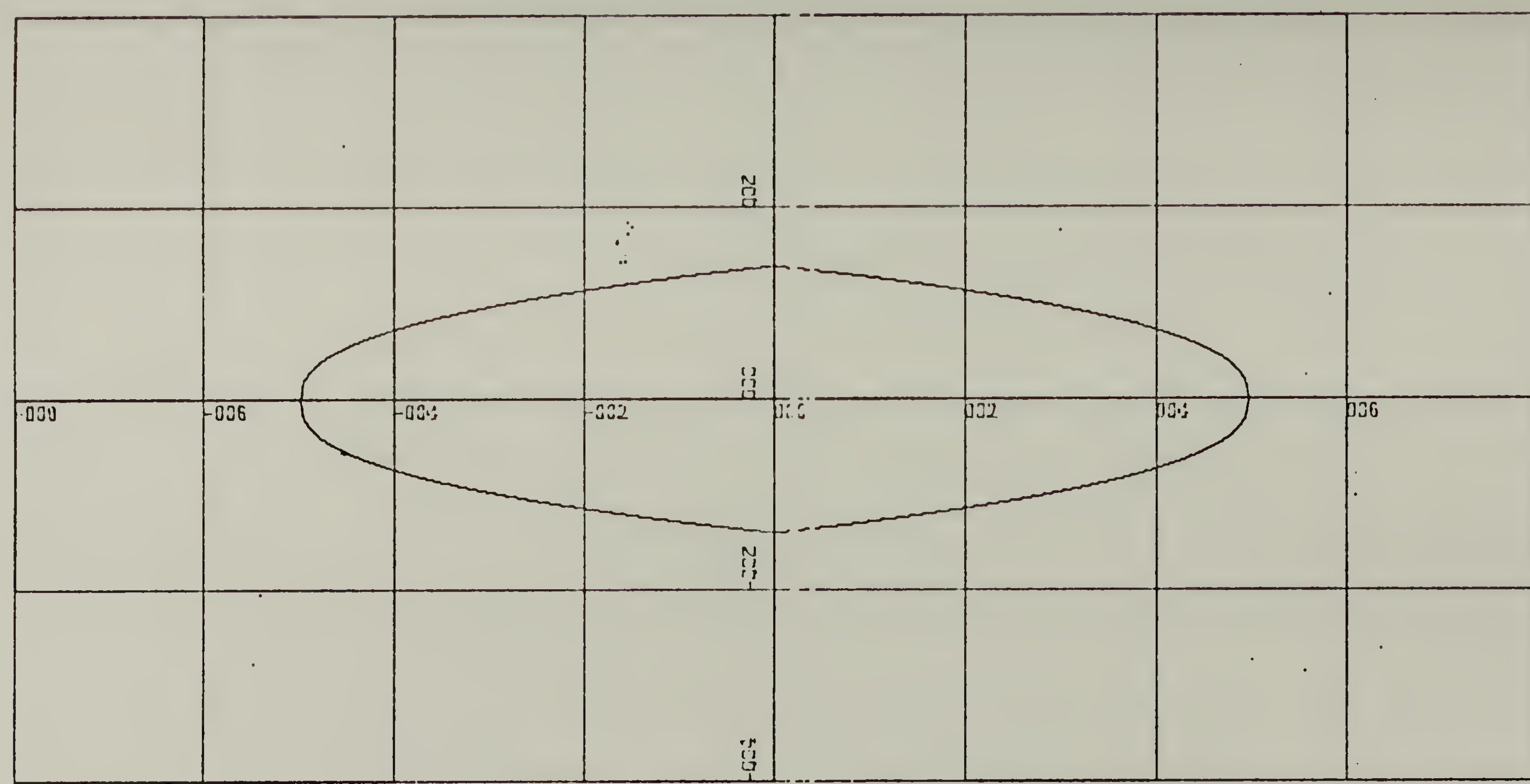


K-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.3

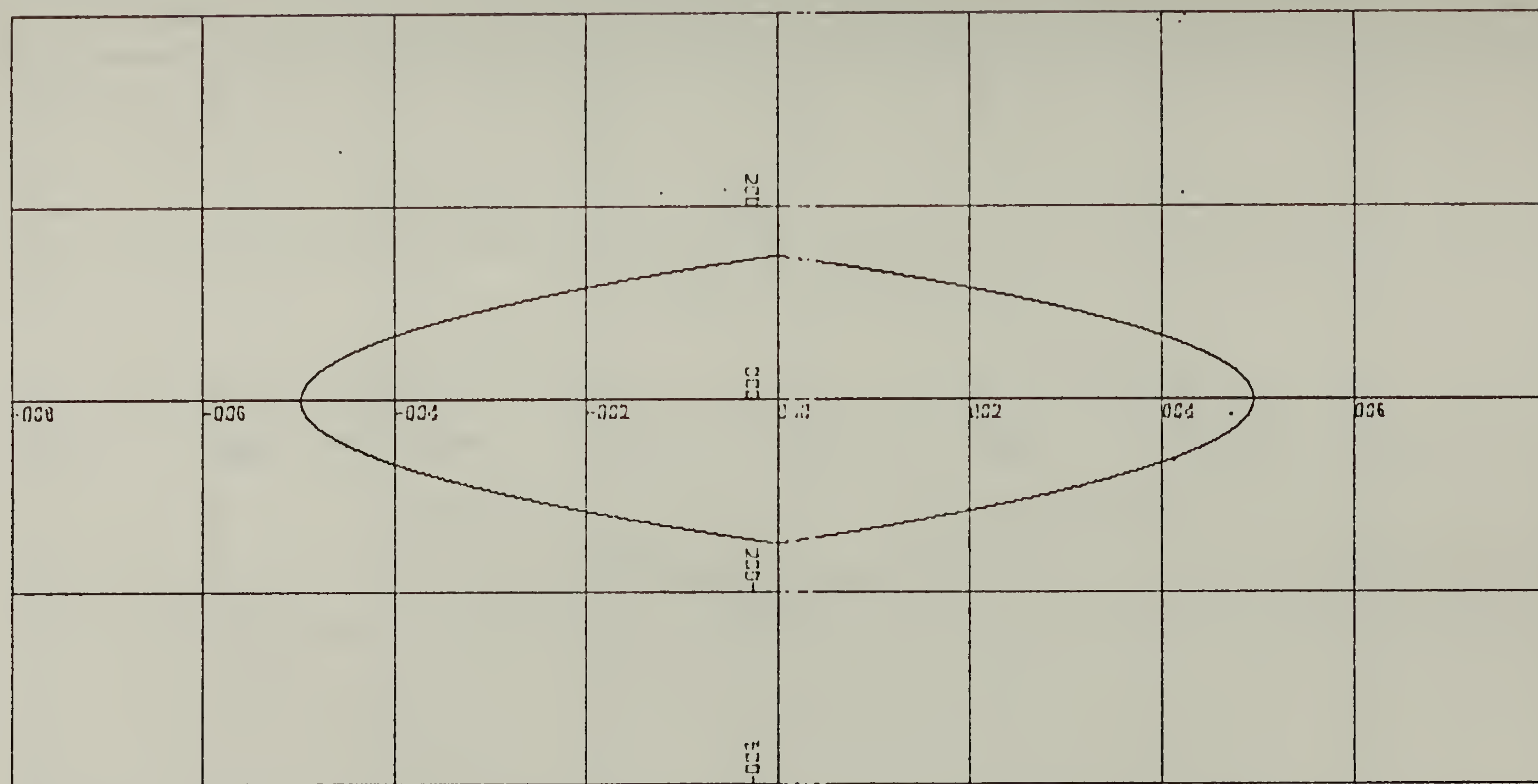
Figure 31. Simply Supported Beam - Case $h=h_o(1-2x/L)^{0.3}$
$P^*=16.827$.



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

RØBERT L. BURNS LCDR, USN, THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.4

Figure 32. Simply Supported Beam - Case $h=h_o(1-2x/L)^{0.4}$
$P^*=16.212$.

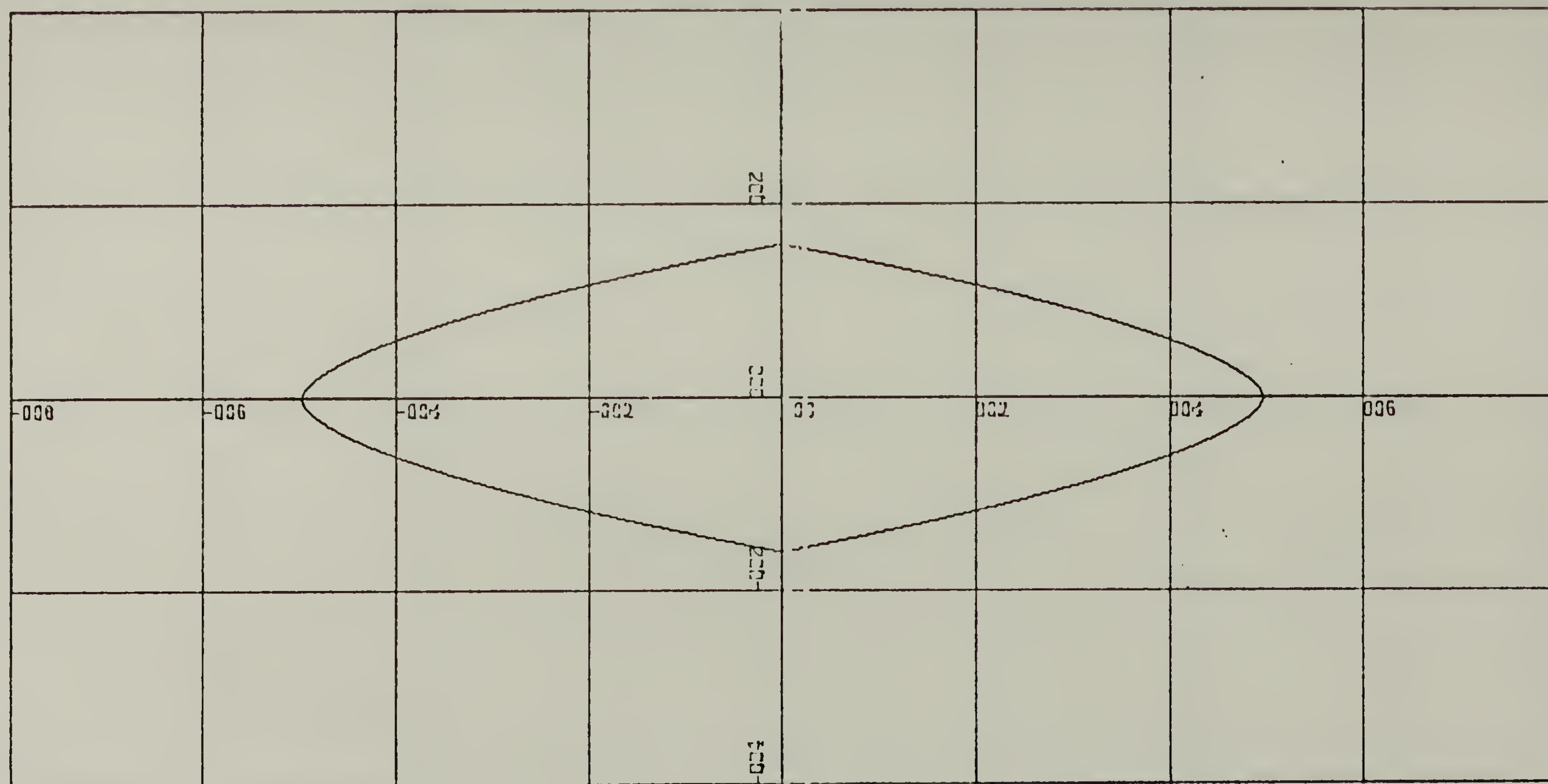


K-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

RØBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.5

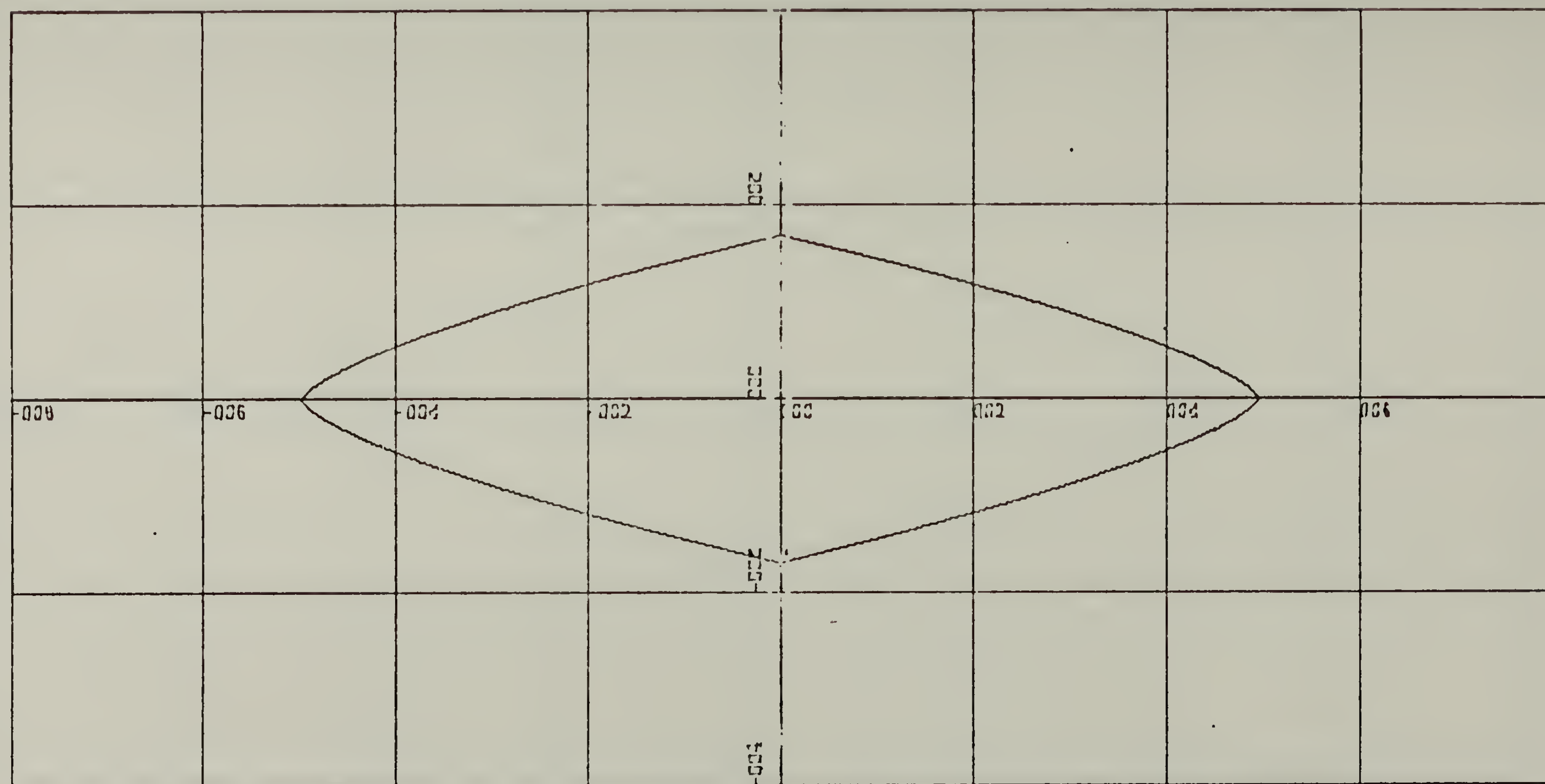
Figure 33. Simply Supported Beam - Case $h=h_o(1-2x/L)^{0.5}$
$P^* = 15.283.$



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.6

Figure 34. Simply Supported Beam - Case $h = h_o(1-2x/L)^{0.6}$
$P^* = 14.013$.

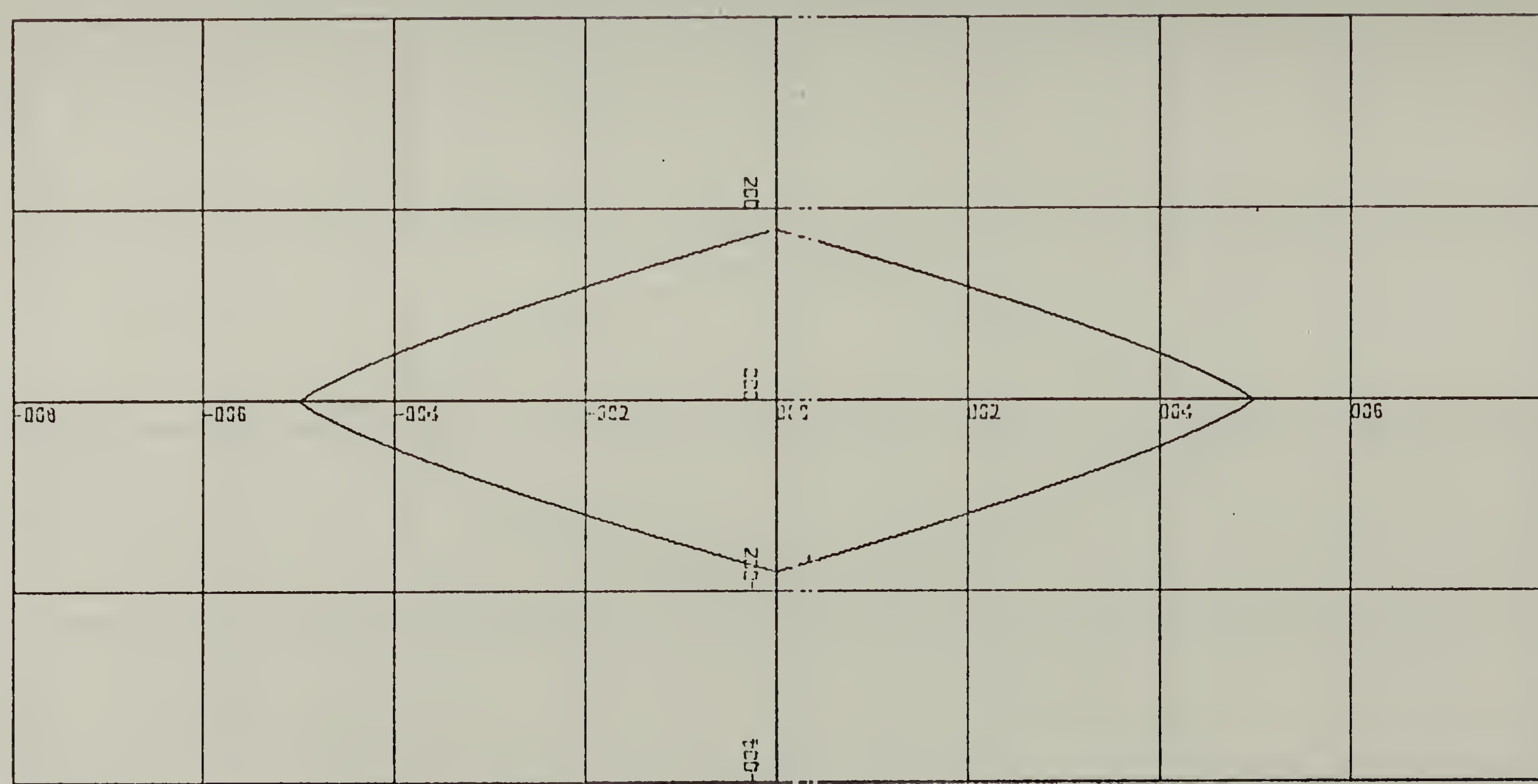


K-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.7

Figure 35. Simply Supported Beam - Case $h=h_o(1-2x/L)^{0.7}$
$P^*=12.347.$



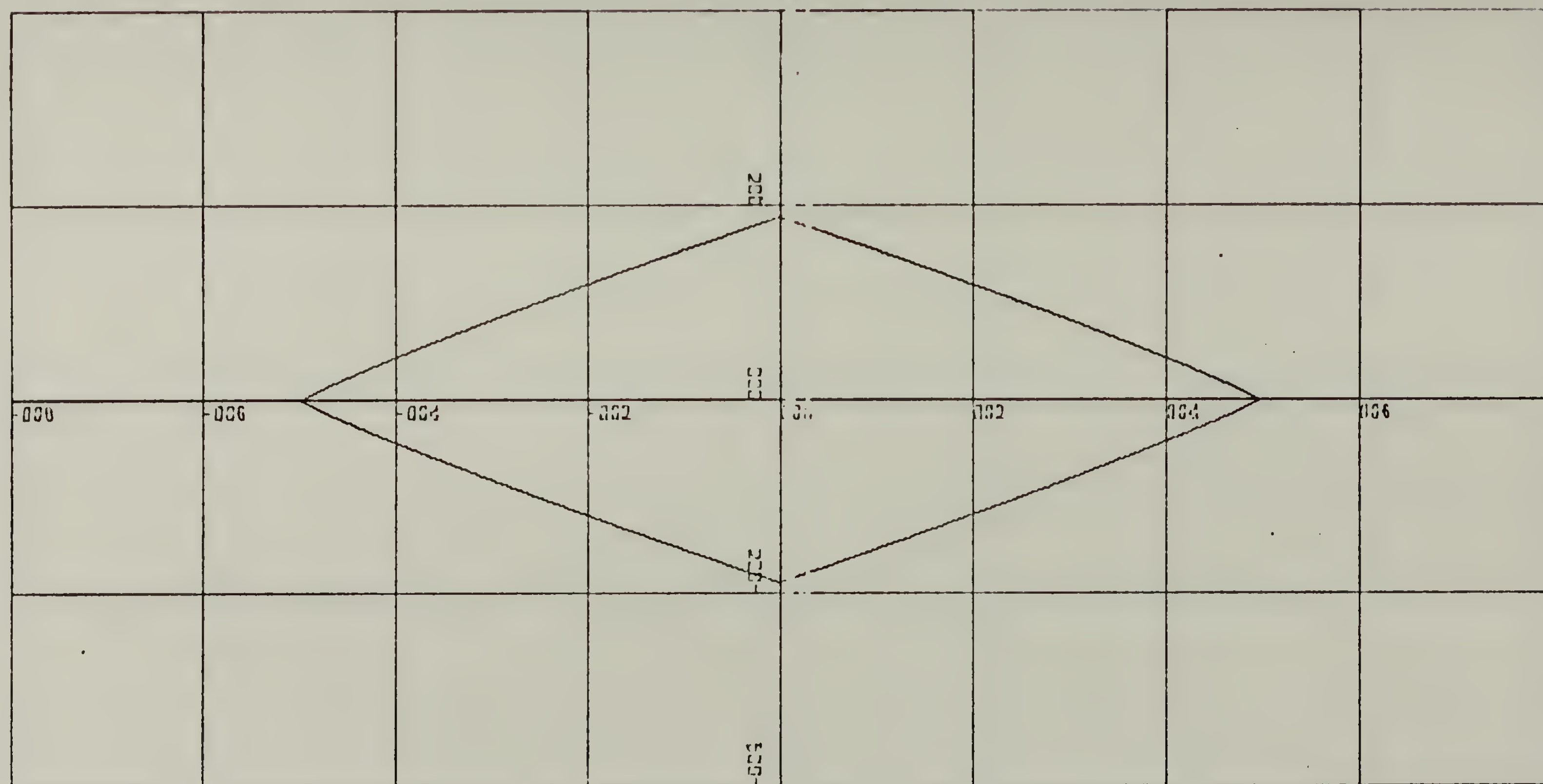
X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.8

Figure 36. 'Simply Supported Beam - Case $h=h_o(1-2x/L)^{0.8}$

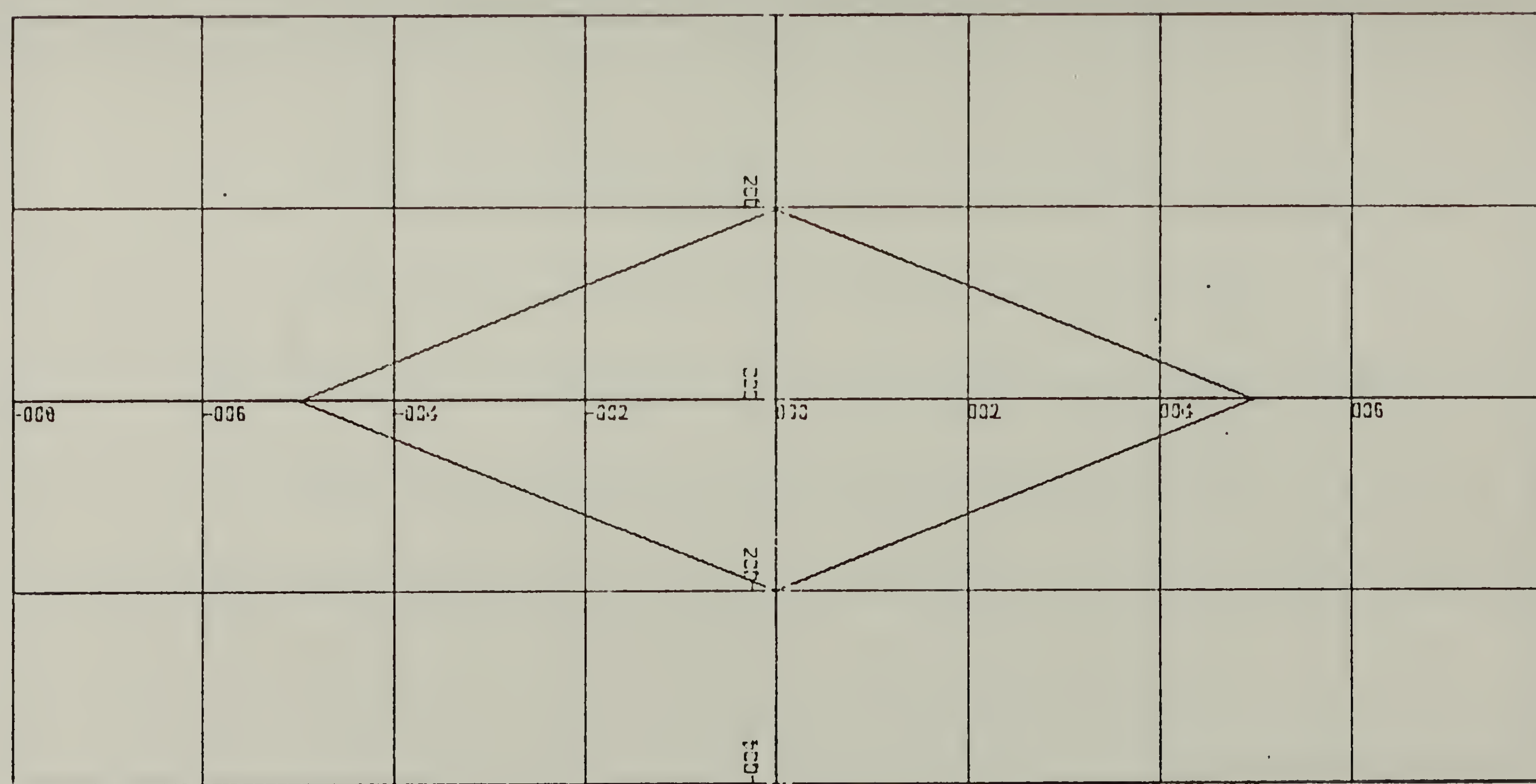
P*=10.181.



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR. USN. THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.9

Figure 37. Simply Supported Beam - Case $h=h_o(1-2x/L)^{0.9}$
$P^*=7.210$.

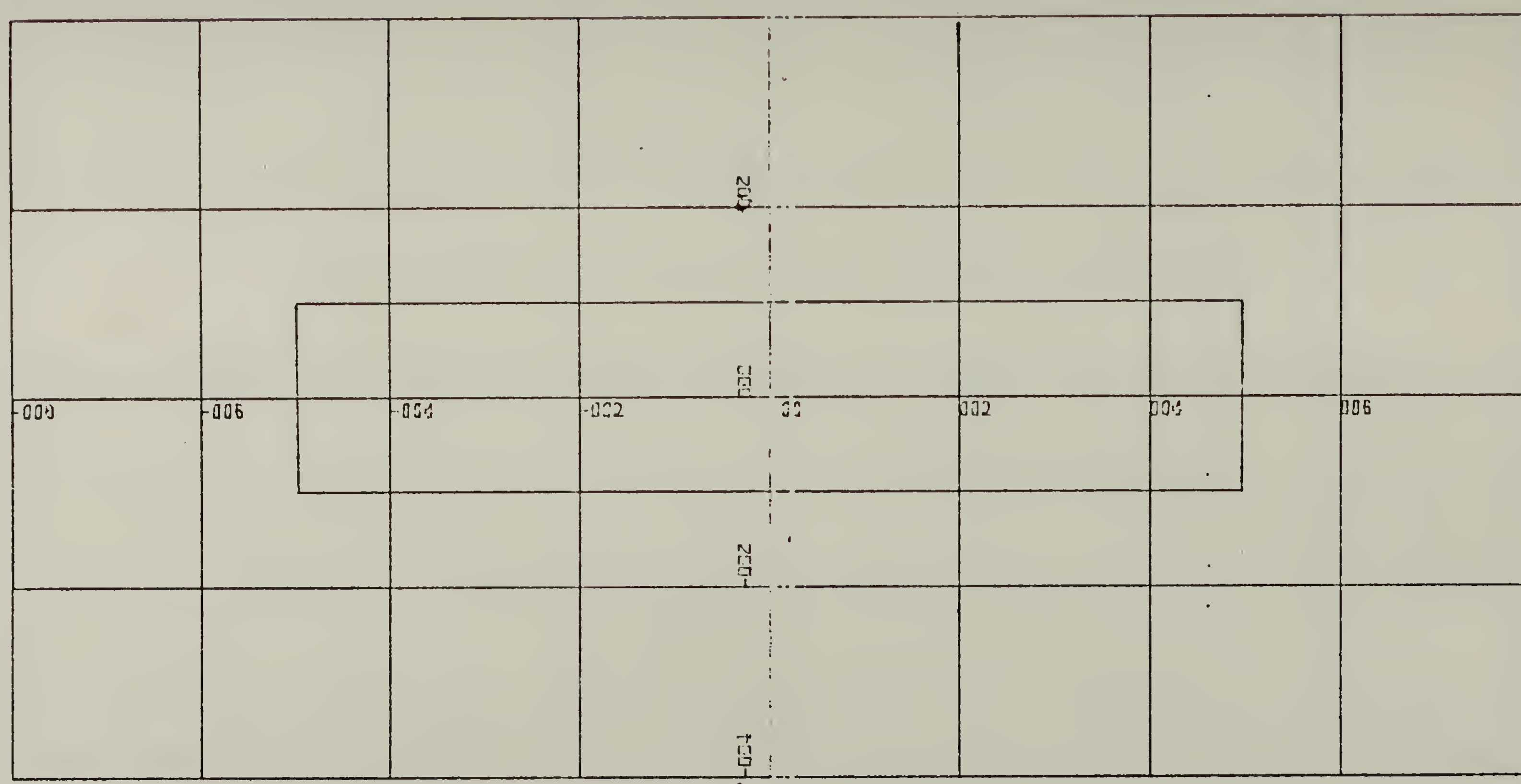


X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=1

Figure 38. Simply Supported Beam - Case $h=h_o(1-2x/L)^{1.0}$, $P^*=0$.



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0

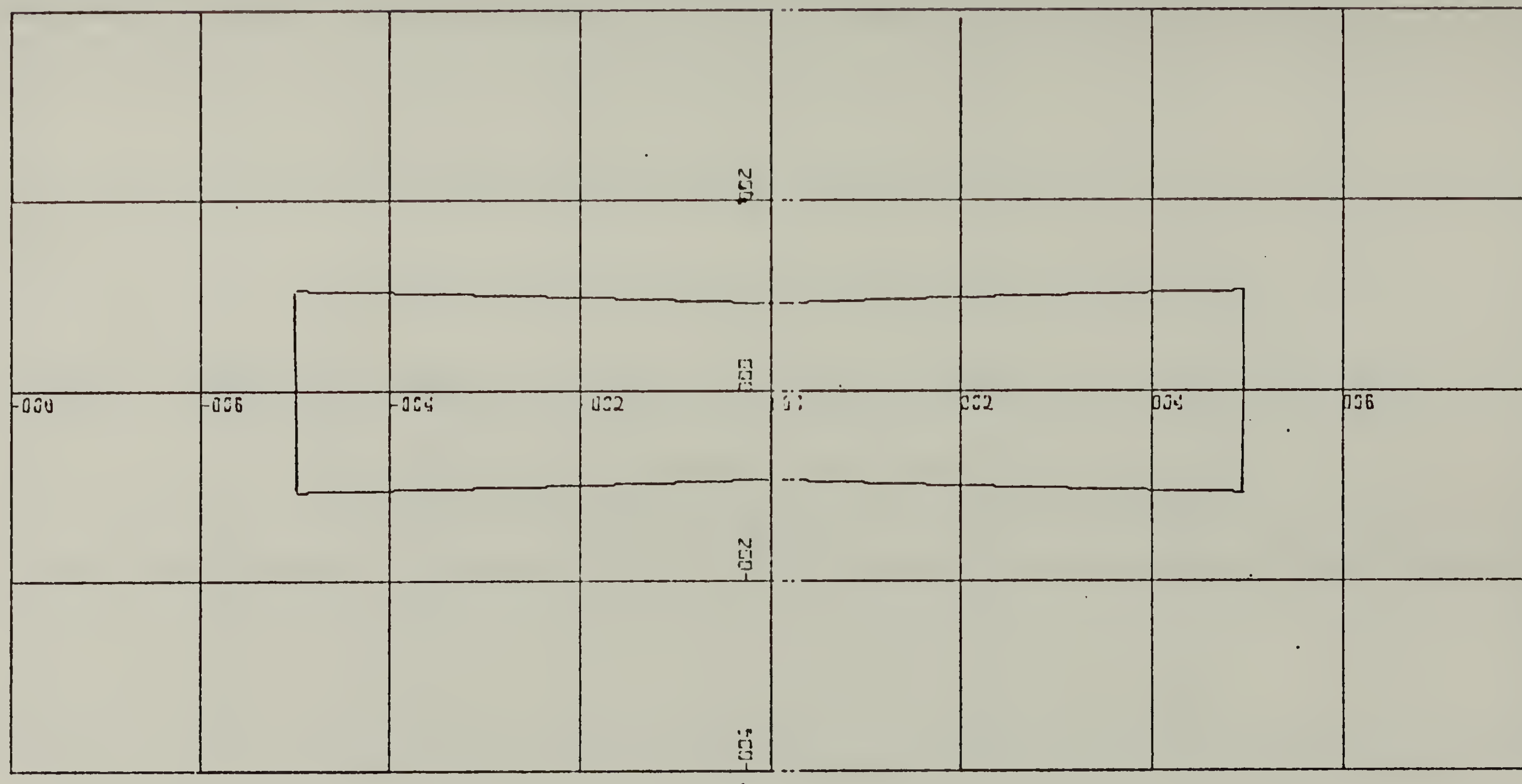
Figure 39. Simply Supported Beam - Case $h=h_o(1+2x/L)^0$, P*=16.936.

X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR. USN. THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.1

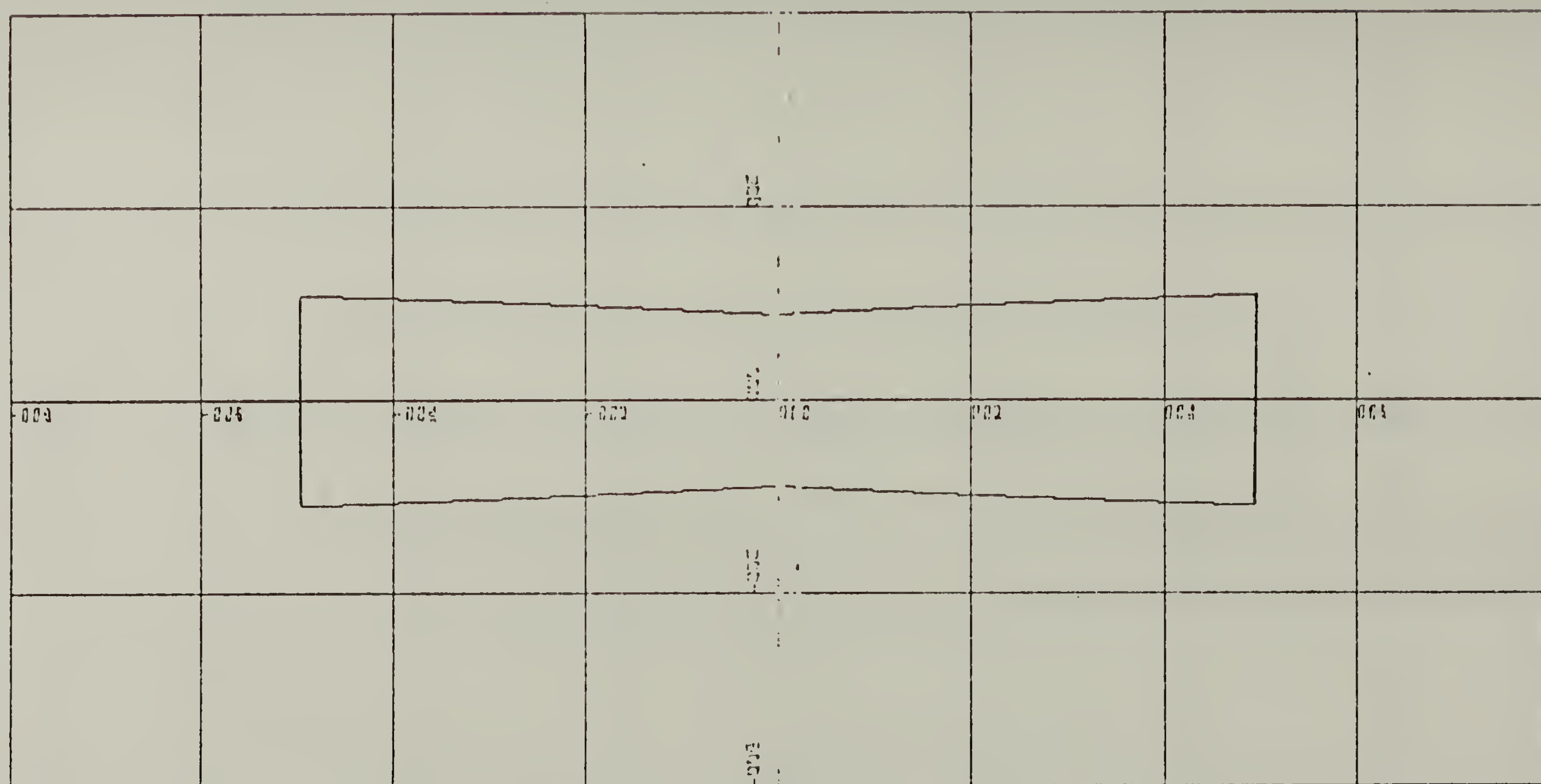
Figure 40. Simply Supported Beam - Case $h\text{-}h_o(1+2x/L)^{0.1}$, $P^*=15.023$.



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.2

Figure 41. Simply Supported Beam - Case $h=h_o(1+2x/L)^{0.2}$, P*=13.215.

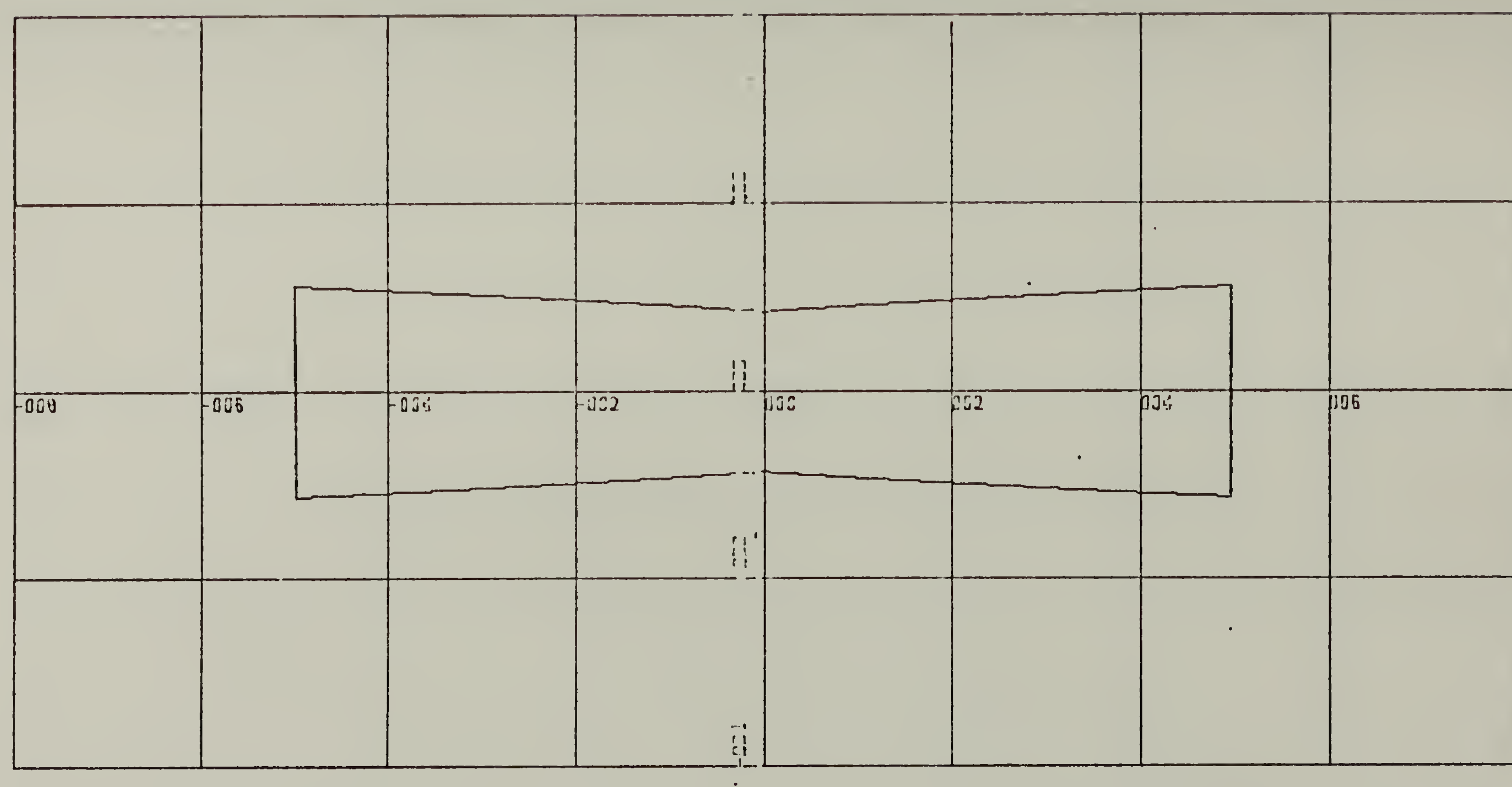


X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.3

Figure 42. Simply Supported Beam - Case $h=h_o(1+2x/L)^{0.3}$, P*=11.508.

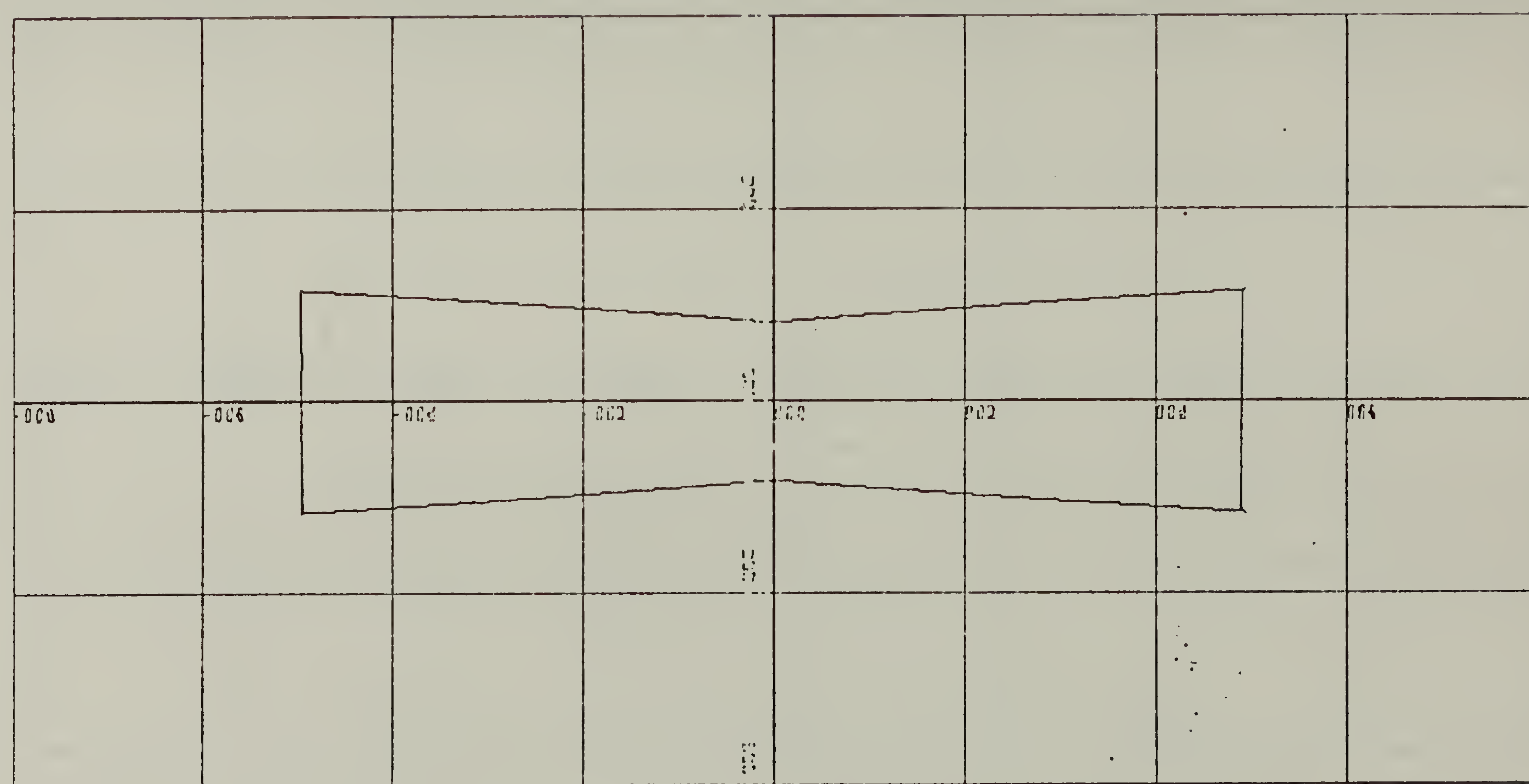


X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.4

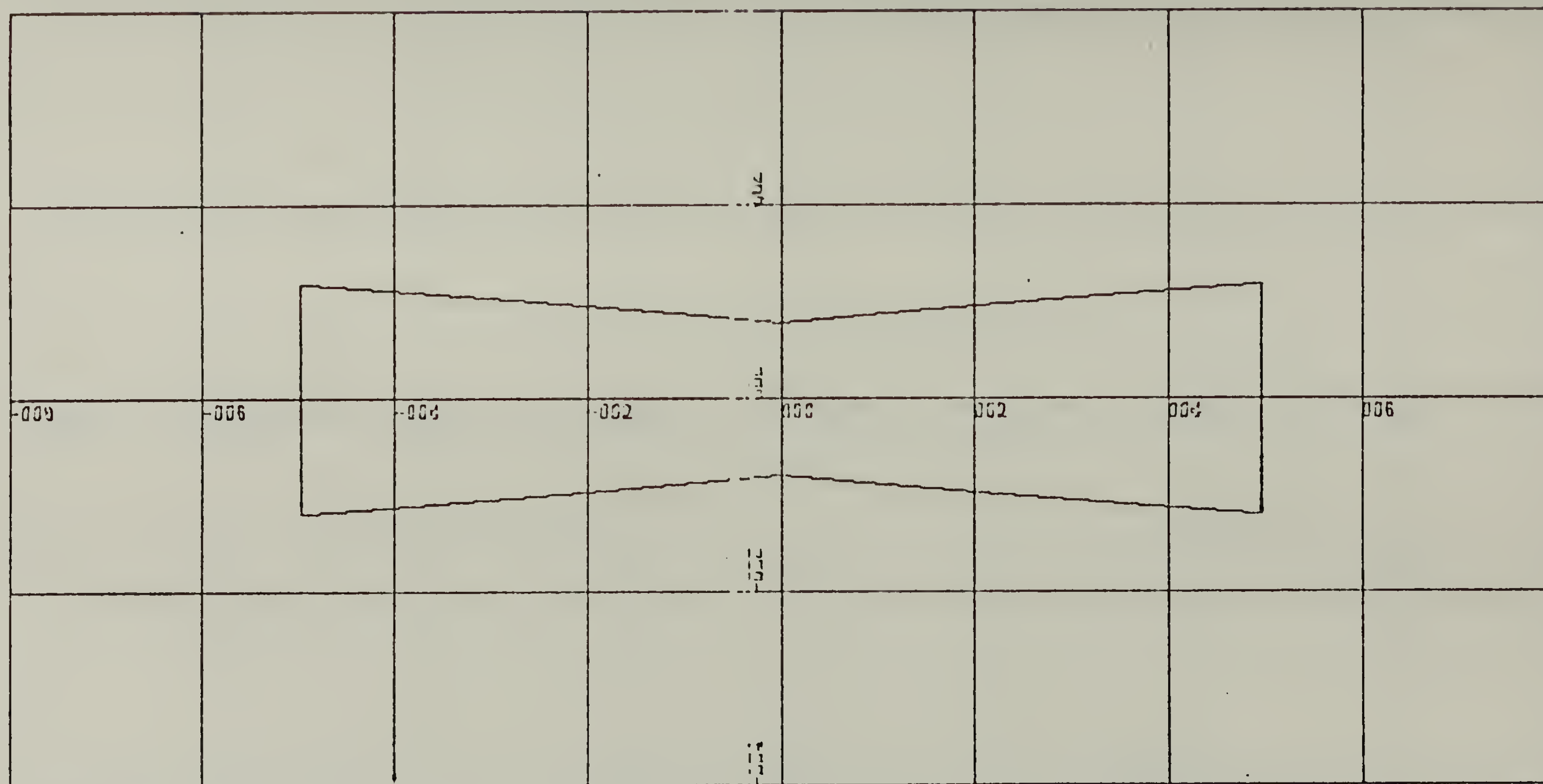
Figure 43. Simply Supported Beam - Case $h=h_o(1+2x/L)^{0.4}$, P*=9.891.



K-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.5

Figure 44. Simply Supported Beam - Case $h=h_o(1+2x/L)^{0.5}$, P*=8.359.

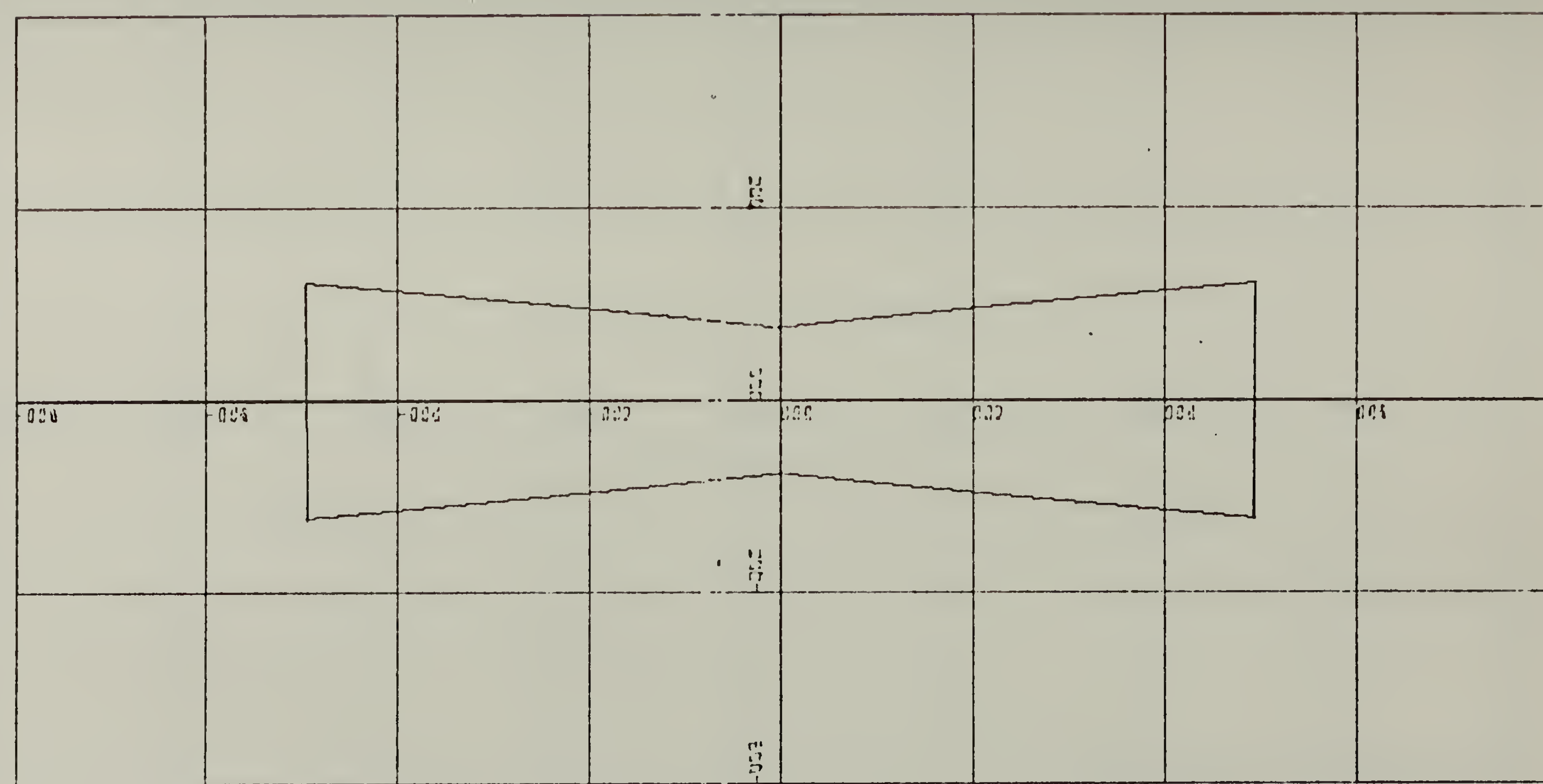


X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.6

Figure 45. Simply Supported Beam - Case $h=h_o(1+2x/L)^{0.6}$, P*=6.898.

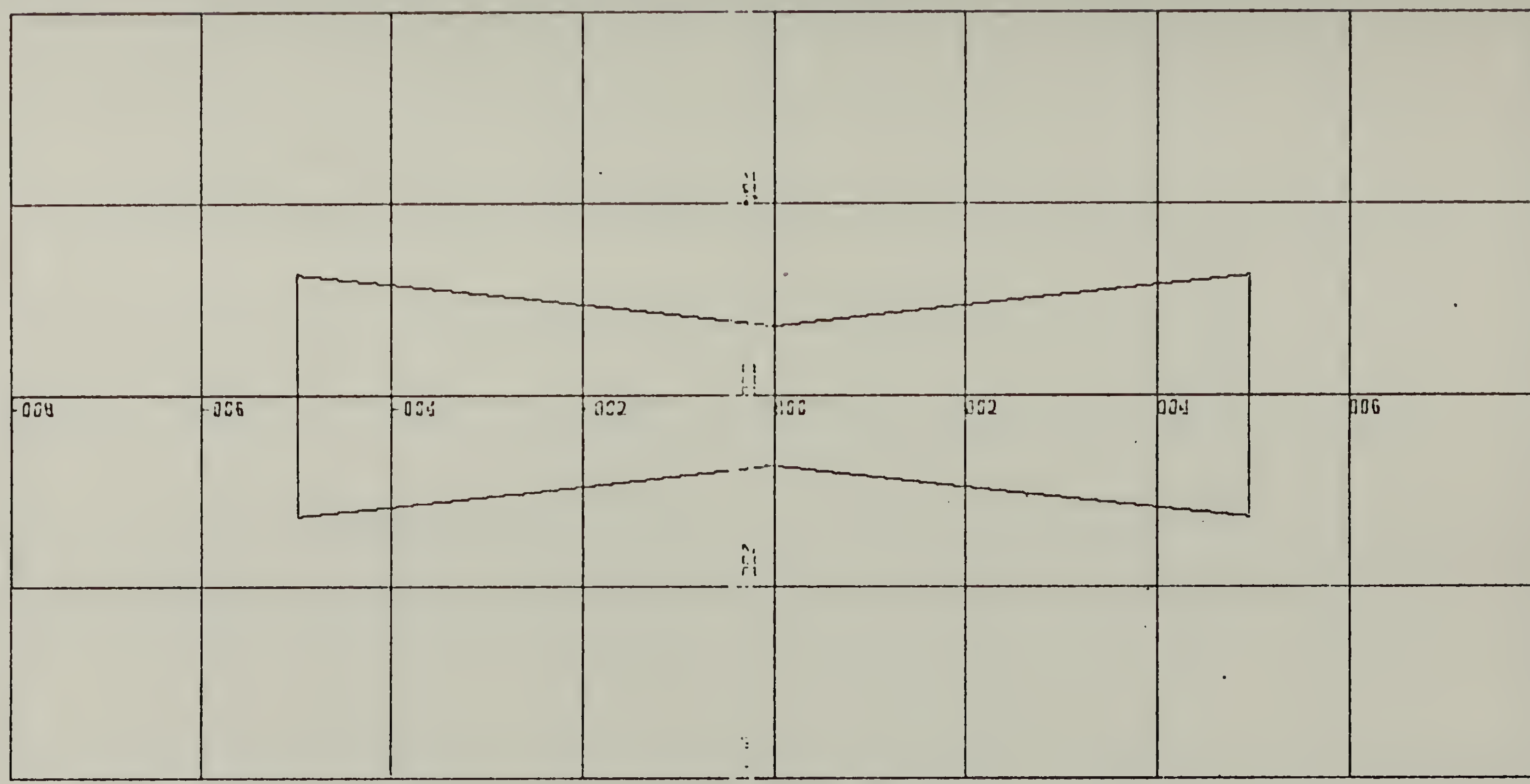


K-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.7

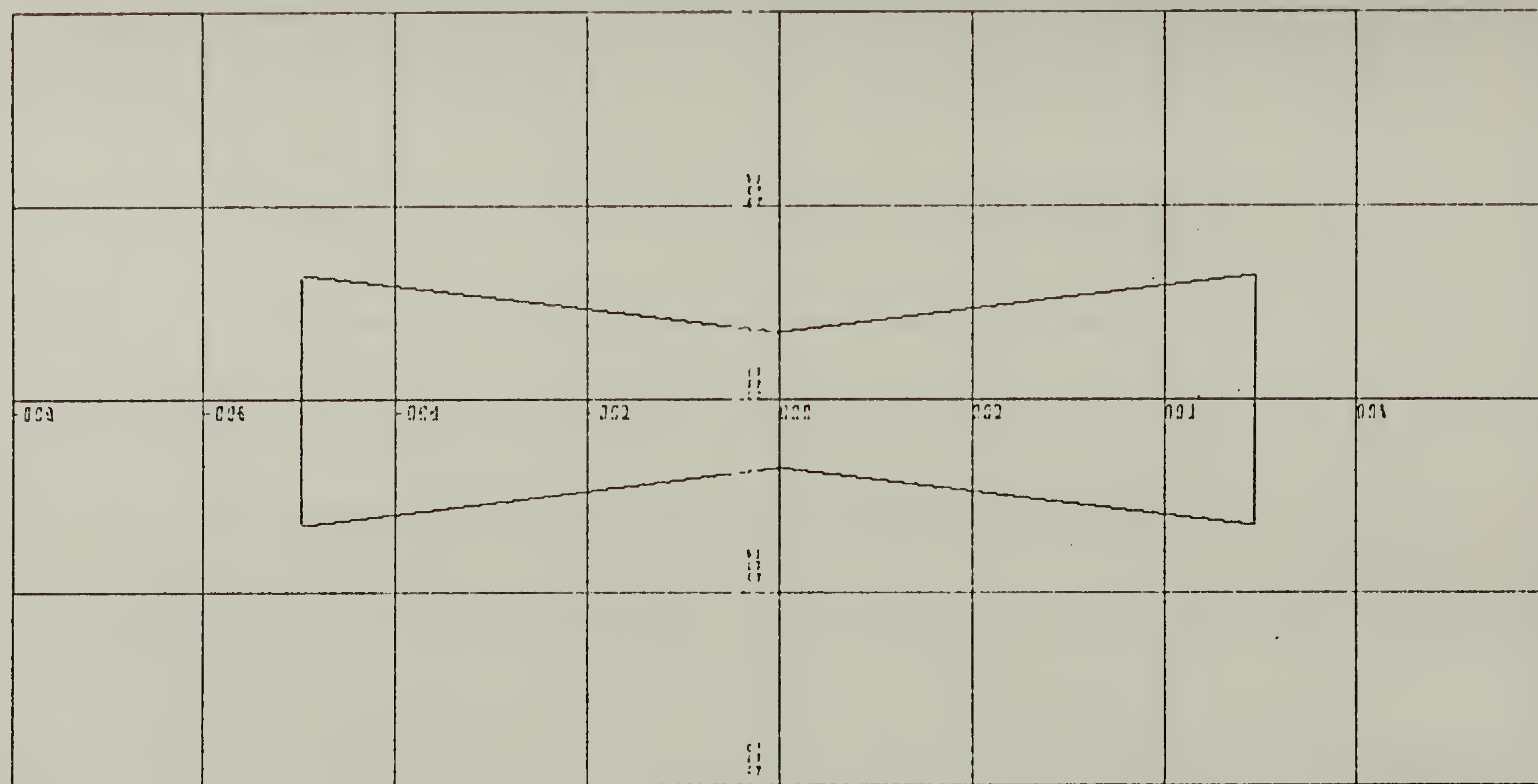
Figure46. Simply Supported Beam - Case $h=h_o(1+2x/L)^{0.7}$, P*=5.490.



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

RØBERT L. BURNS LCDR, USN, THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.8

Figure 47. Simply Supported Beam - Case $h=h_o(1+2x/L)^{0.8}$, $P^*=4.102$.

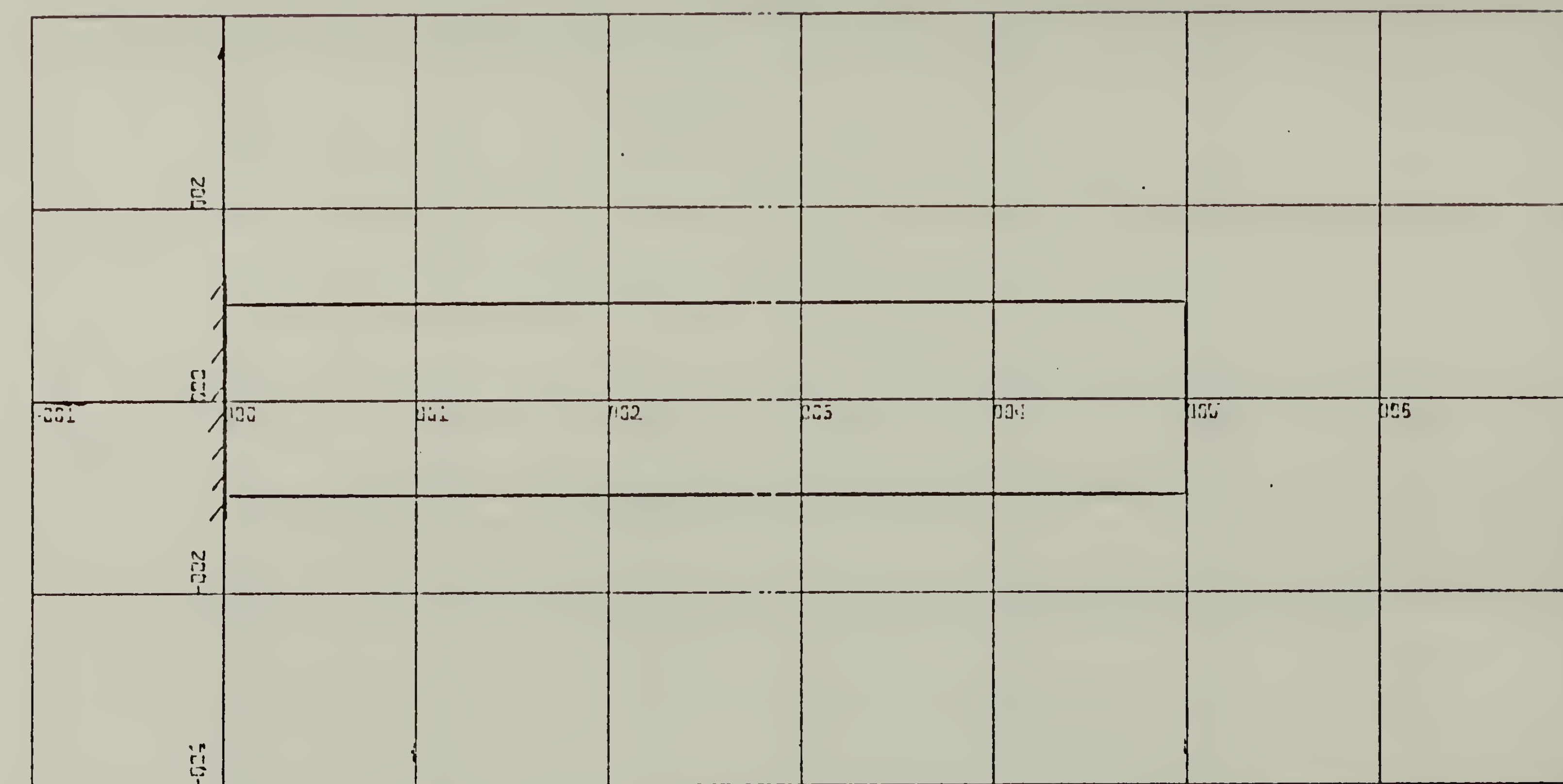


K-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N= 0.9

Figure 48. Simply Supported Beam - Case $h=h_o(1+2x/L)^{0.9}$, P*=2.639.

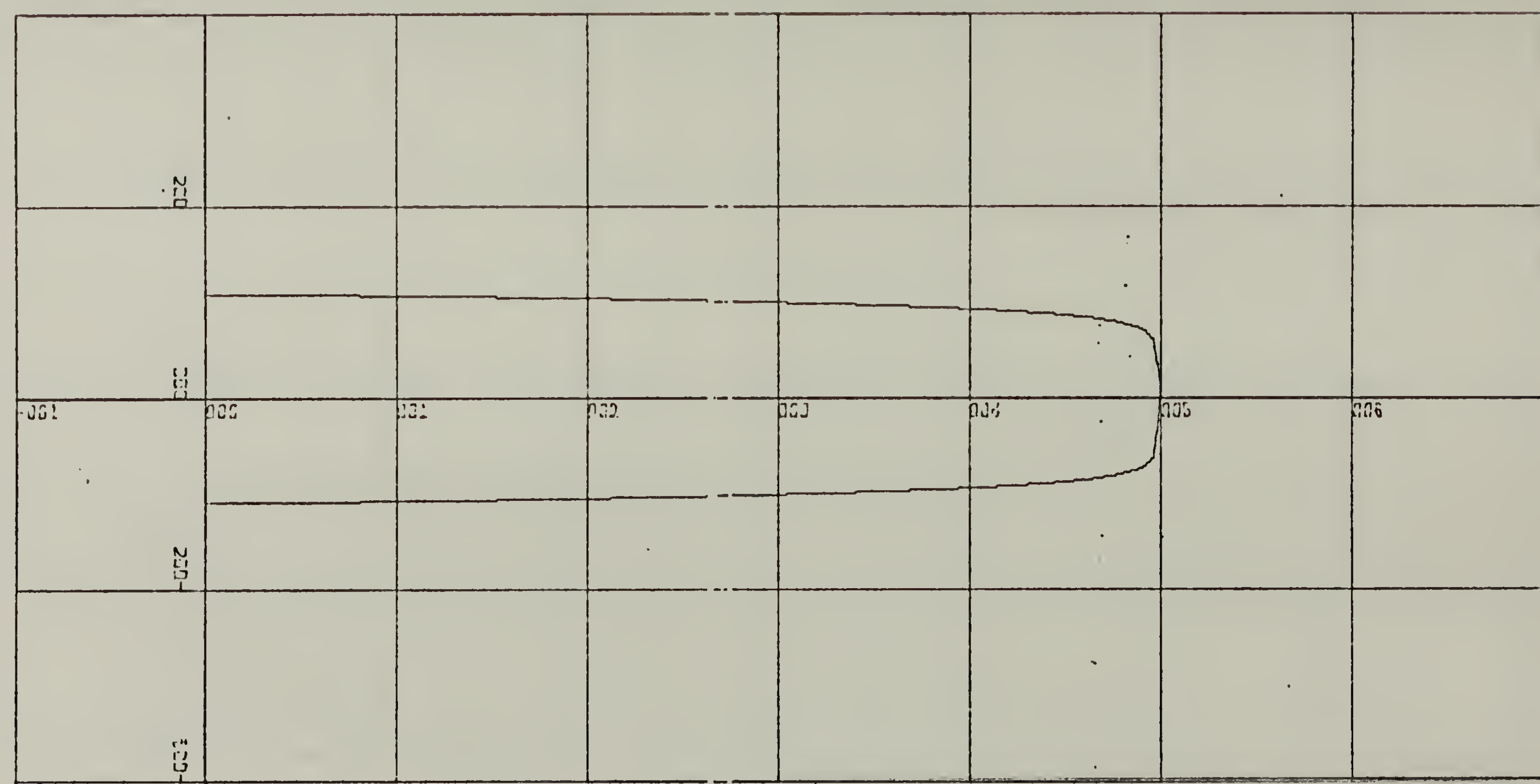


X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N= 0.

Figure 49. Cantilever Beam - Case $h=h_o(1+x/L)^0$, P* = 4,013.

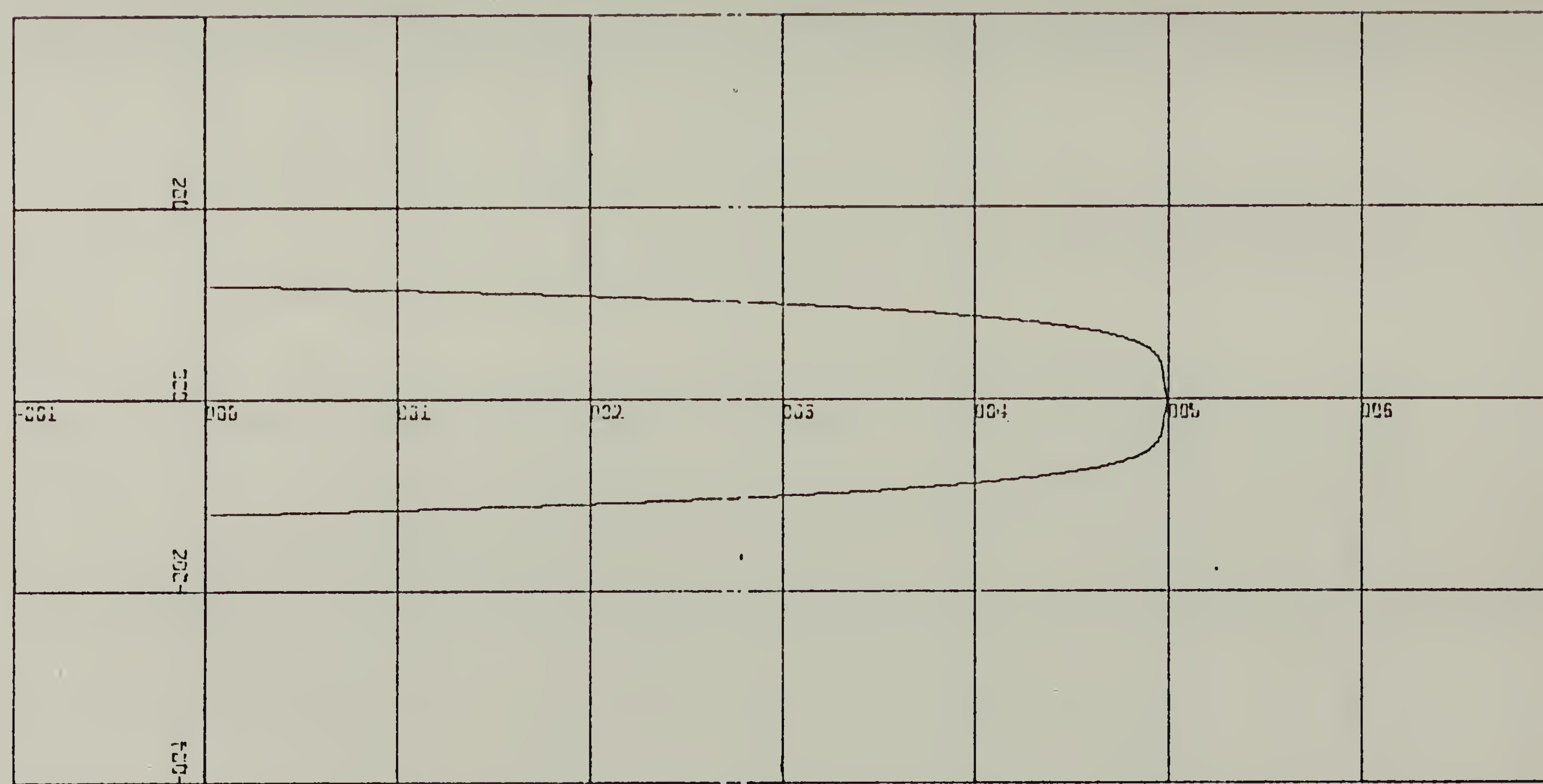


-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.1

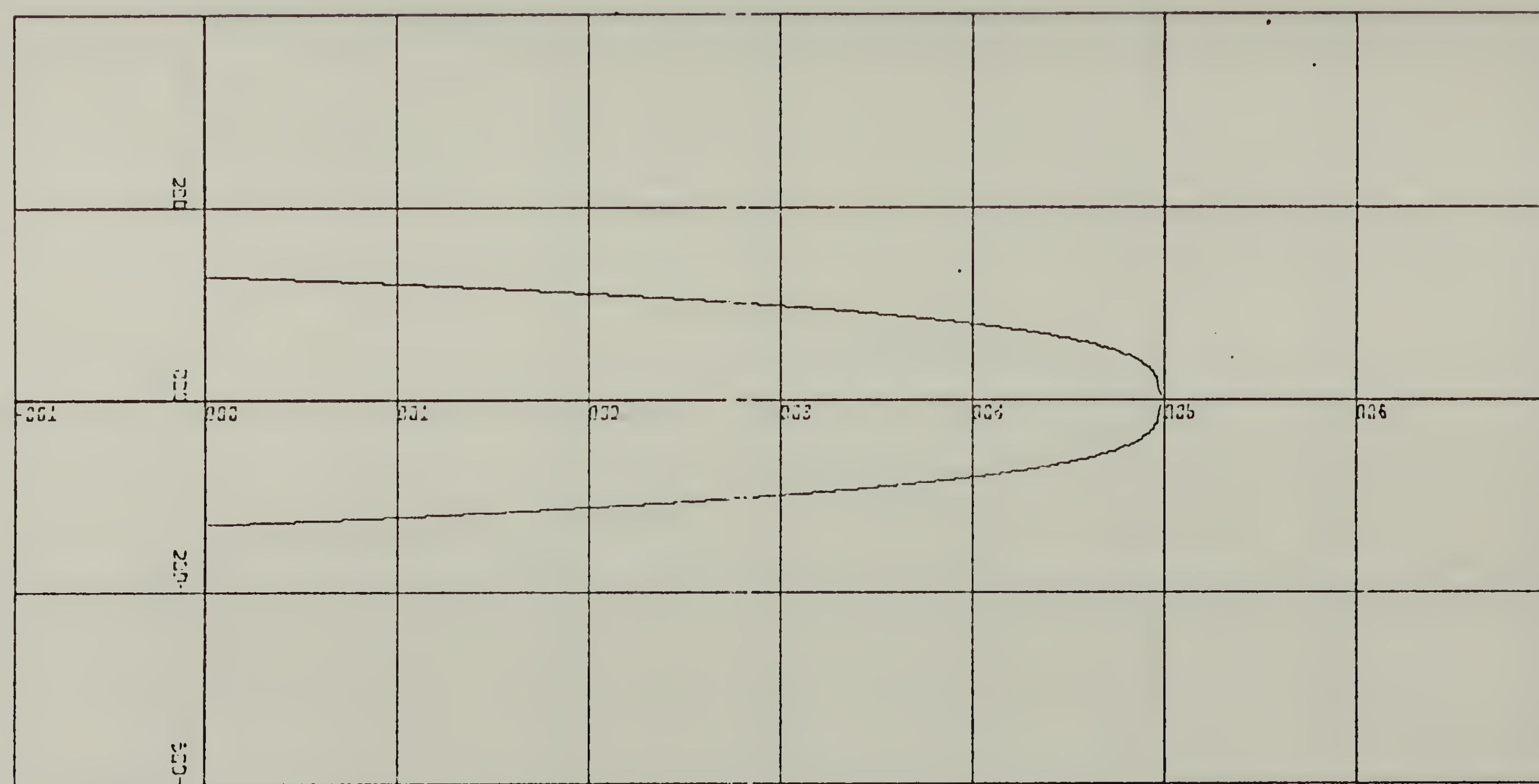
Figure 50. Cantilever Beam - Case $h=h_o(1+x/L)^{0.1}$, $P^*=4.239$



X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.2

Figure 51. Cantilever Beam - Case $h=h_o(1+x/L)^{0.2}$, P*=4.433.

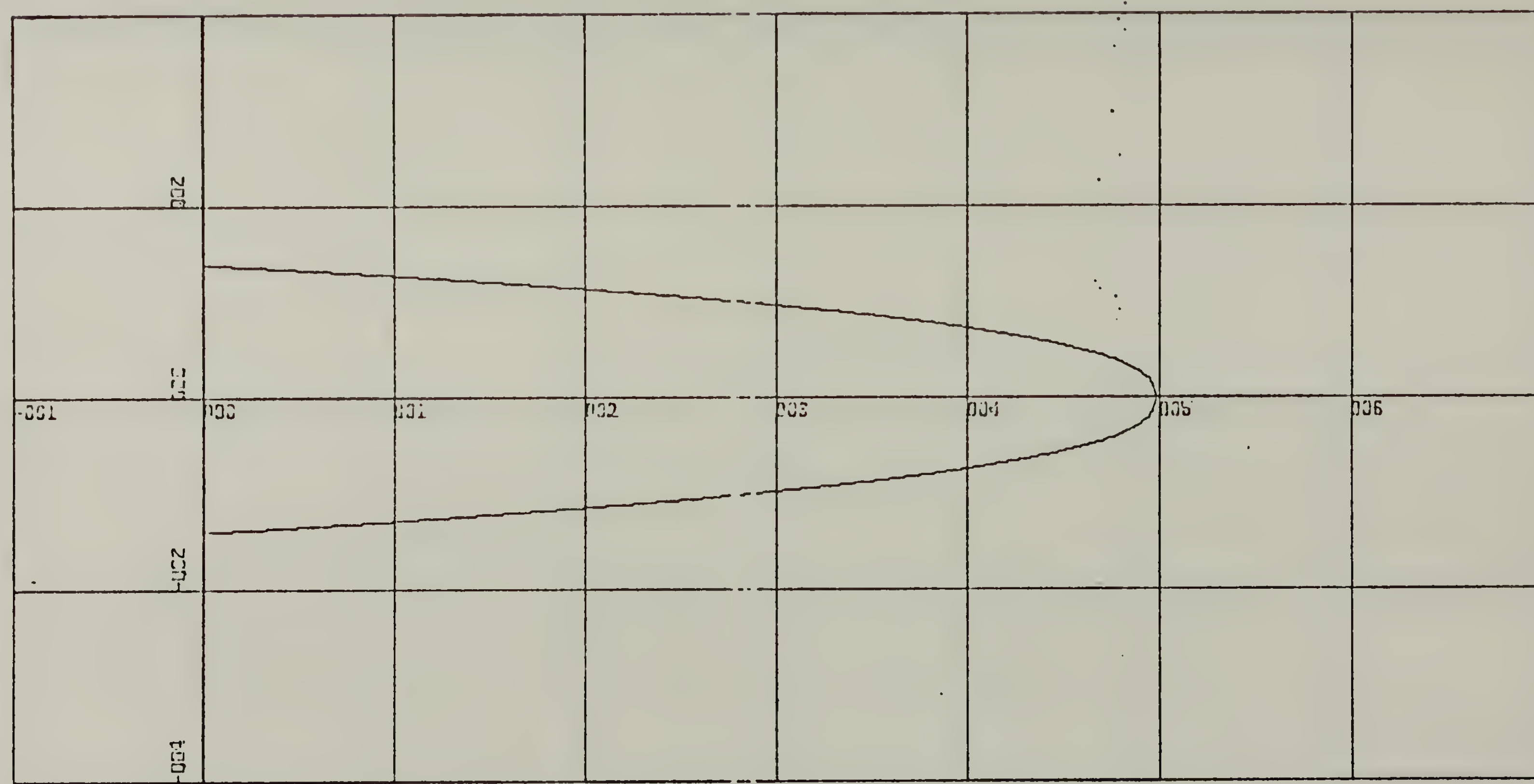


X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

RØBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.3

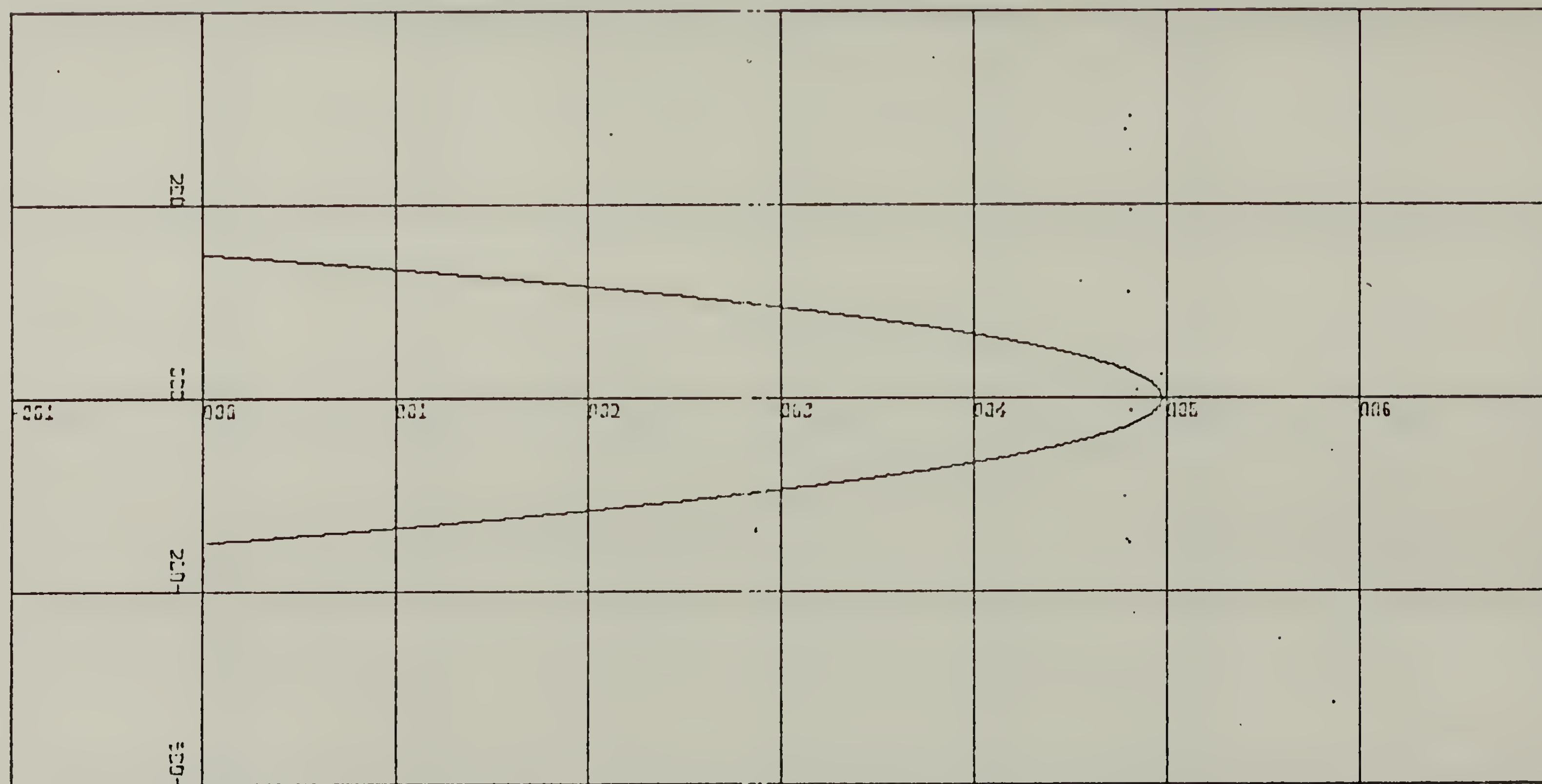
Figure 52. Cantilever Beam - Case $h=h_o(1+x/L)^{0.3}$", P*=4.595.



X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS CDR, USN, THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.4

Figure 53. Cantilever Beam - Case $h=h_o(1+x/L)^{04}$, $P^*=4.724$.

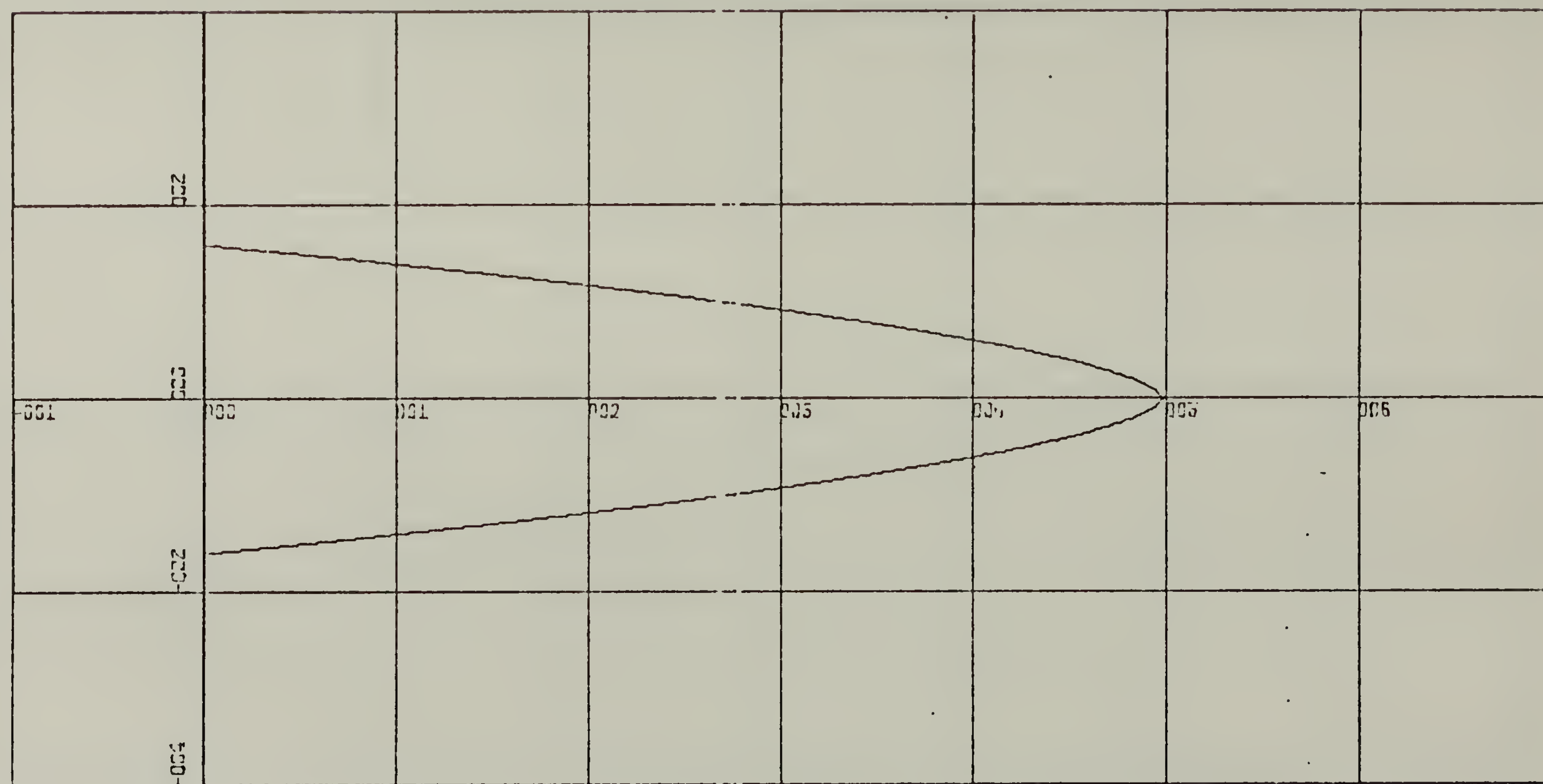


X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

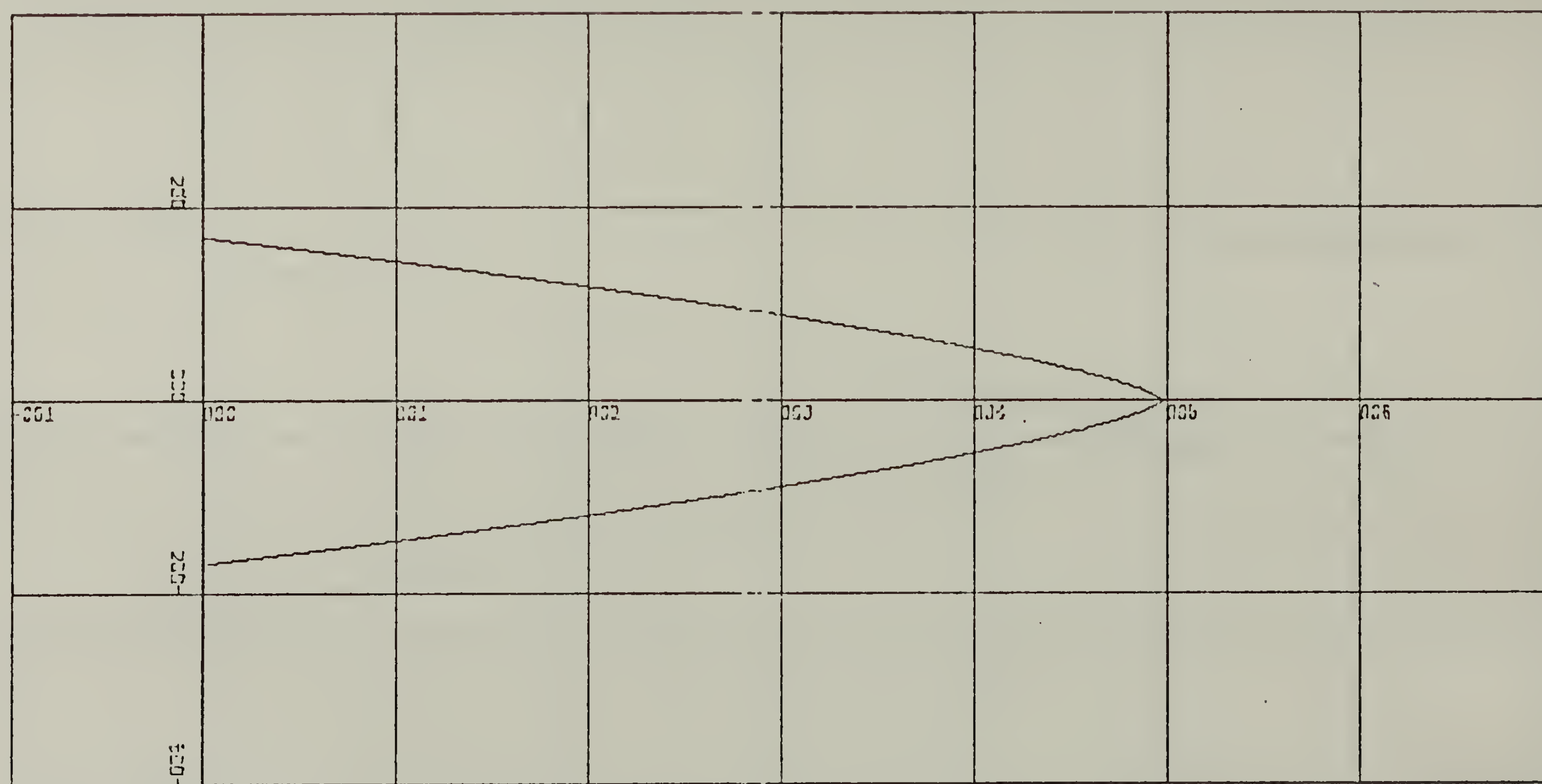
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.5

Figure 54. Cantilever Beam - Case $h=h_o(1+x/L)^{05}$, P*=4.821.



X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.
RØBERT L. BURNS _CDR. USN. THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.6

Figure 55. Cantilever Beam - Case $h=h_o(1+x/L)^{0.6}$, P*=4.885.



K-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS LCDR, USN, THESIS

VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.7

Figure 56. Cantilever B-am - Case $h=h_o(1+x/L)^{0.7}$, P*=4.917.

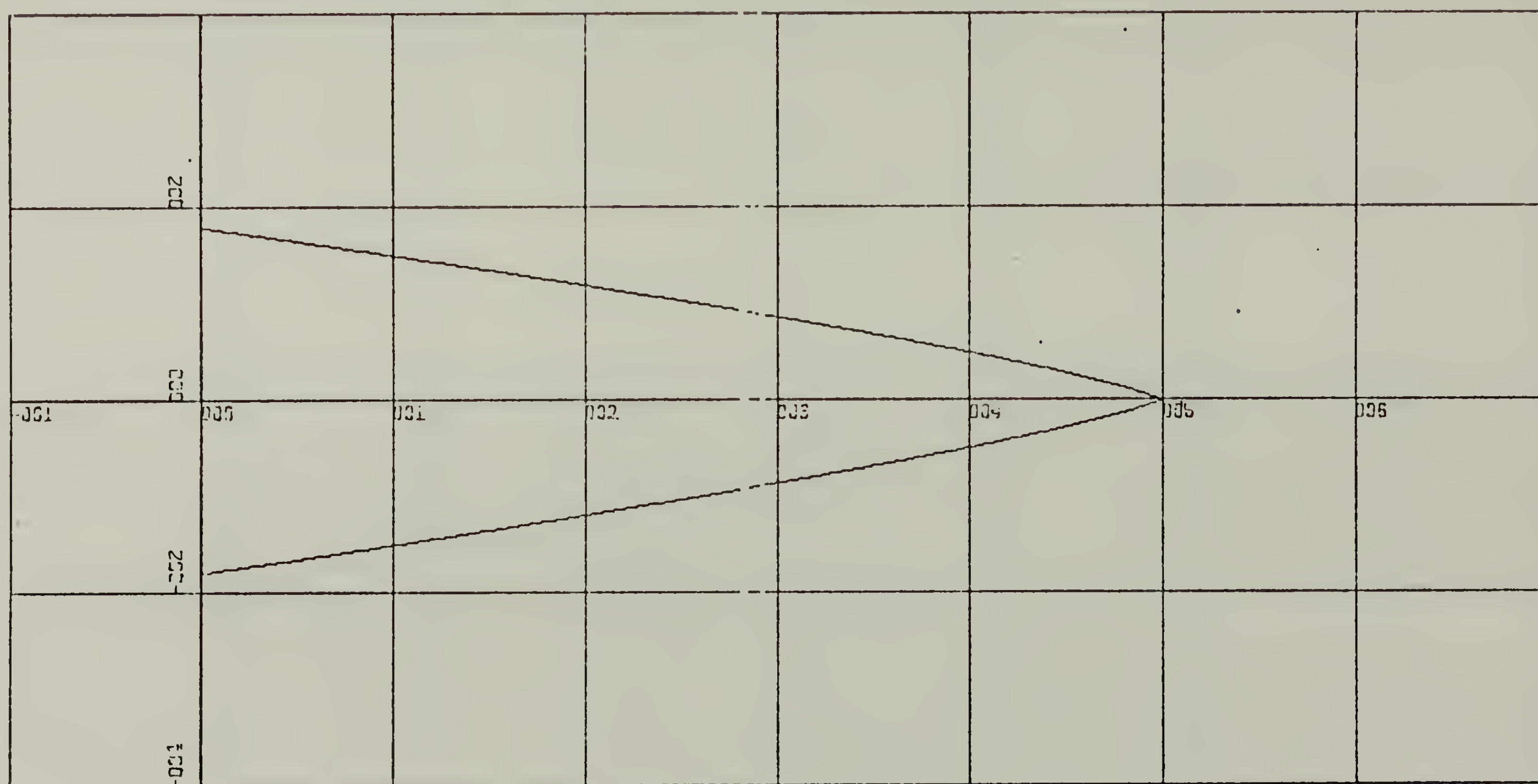


Figure 57. Cantilever Beam - Case $h=h_o(1+x/L)^{0.8}$, P*=4.915.

X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.
RØBERT L. BURNS LCDR, USN, THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.8

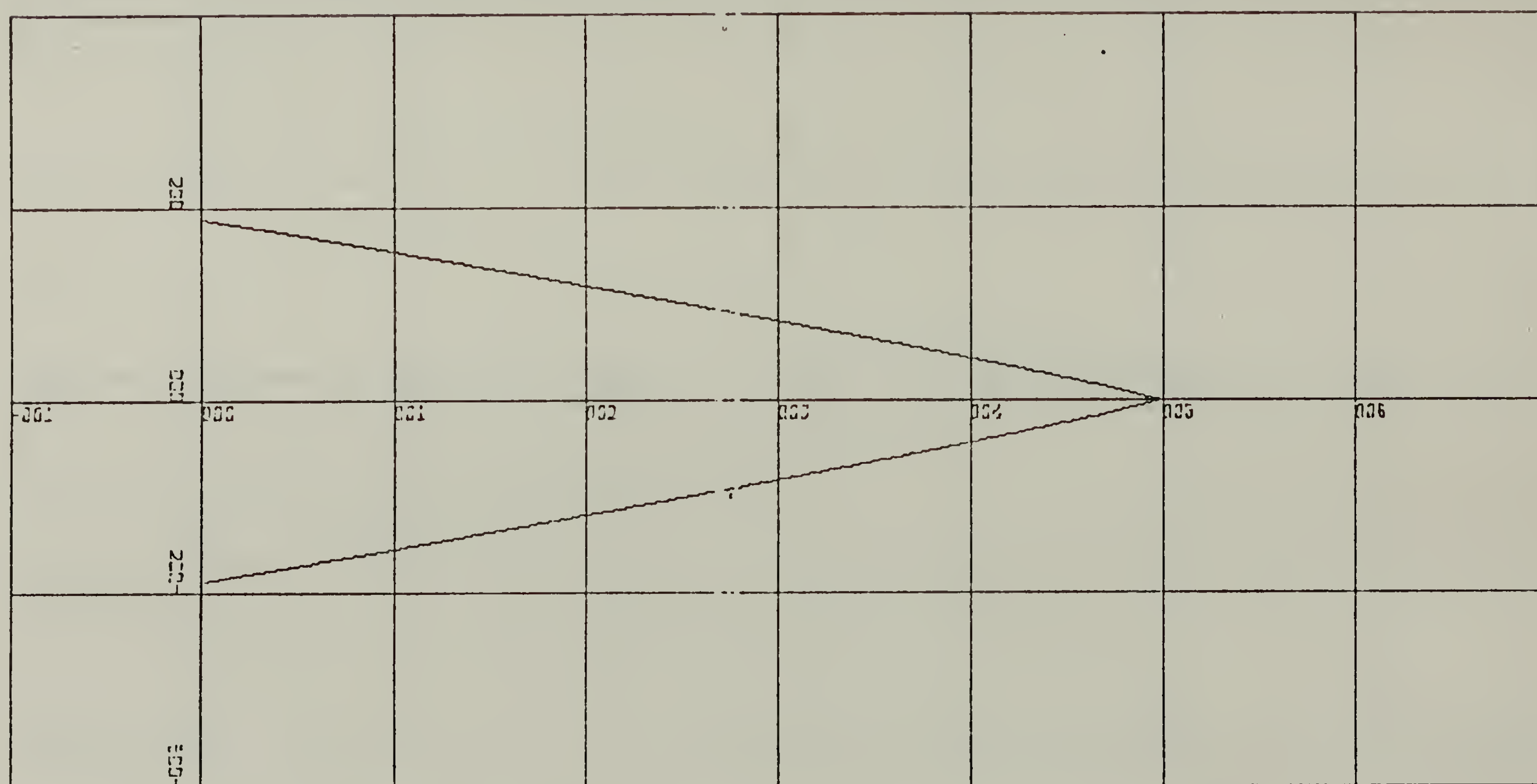


Figure 58. Cantilever Beam - Case $h=h_o(1+x/L)^{0.9}$, P*=4.880.

X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.

ROBERT L. BURNS CDR, USN, THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=0.9

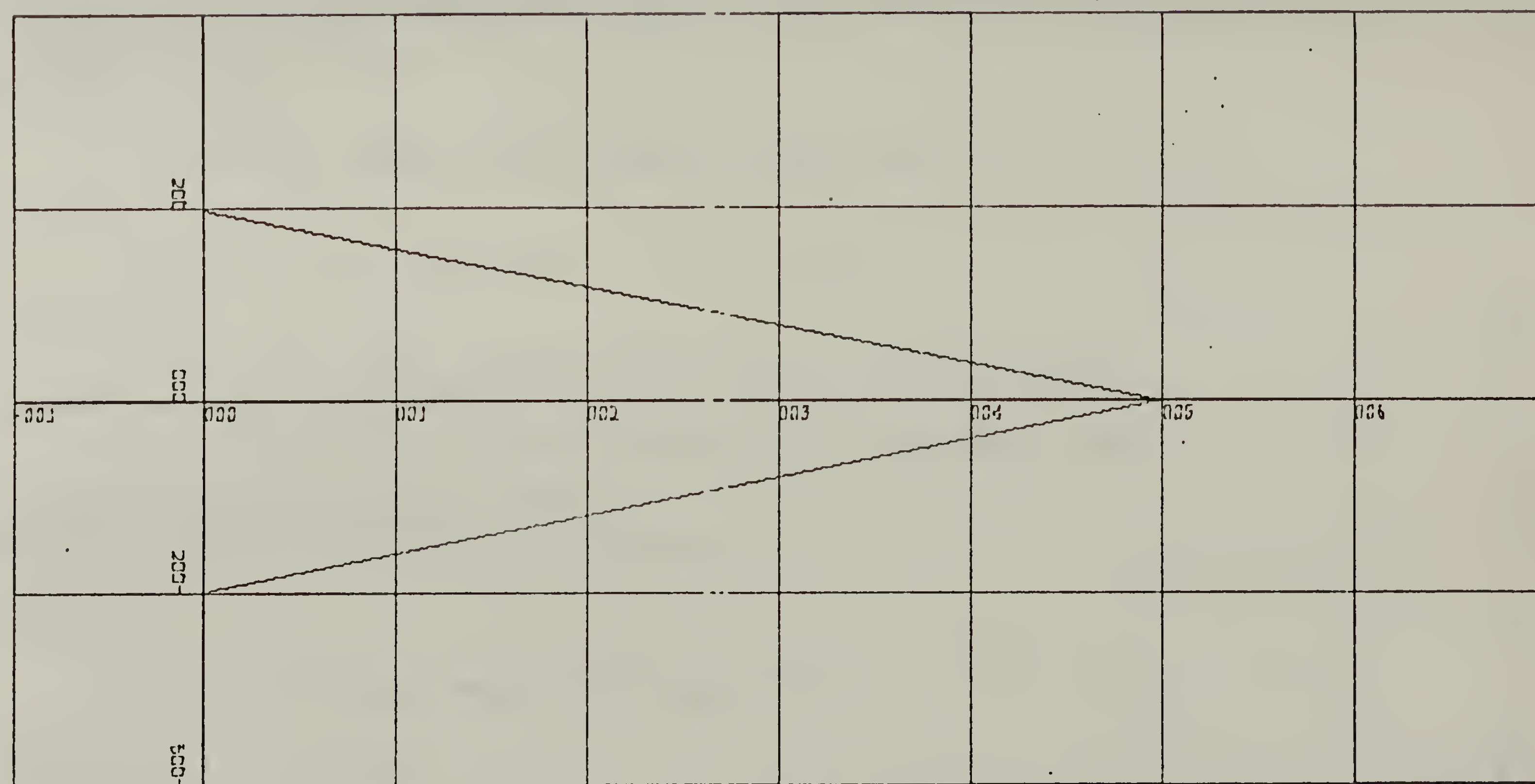


Figure 59. Cantilever Beam - Case $h=h_o(1+x/L)^{1.0}$, P*=4.810.

X-SCALE=1.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.
RØBERT L. BURNS LCDR. USN. THESIS
VARIABLE HEIGHT BEAM DESIGN -- CASE N=1.0

COMPUTER PROGRAMS

1. BESSEL EQUATION CURVE GENERATION

```
      DIMENSION AXIS(601),FJX(601),FJX2(601),PCRIT(601),POPT(601)
      REAL LABEL1/4HFJX2/,LABEL2/4H FJX/
      REAL*8 ITITLE(12)
      READ(5,2000) ITITLE
 2000 FORMAT (6A8)
      DO 100 I=1,10
C
C        CALCULATE THE VALUE OF THE POWER N WHERE N=AI
C
      AI=I-1
C
C        CHANGE THE THE POWER VALUE TO NON-INTEGER
C
      AI=0.1*AI
C
C        CALCULATE THE VALUE OF THE FIRST BESSEL FUNCTION ORDER
C
      ANU=(1.0-AI)/(4.0-2.0*AI)
C
C        CALCULATE THE VALUE OF THE SECOND BESSEL FUNCTION ORDER
C
      ANU2=(5.0-3.0*AI)/(4.0-2.0*AI)
      WRITE (6,500) ANU,ANU2
  500 FORMAT (////' ',10X,F10.4,10X,F10.4/'///)
      X=0.0
      K=0
   10 K=K+1
      X=X+0.01
C
C        GENERATE A STOWAGE ARRAY FOR VALUES OF X
C
      AXIS(K)=X
      CALL SUM (X,ANU,FUN)
      EXPX=ANU
C
C        FIRST BESSEL FUNCTION OF ARGUMENT K
C
      FJX(K)=FUN*X**EXPX
      CALL SUM (X,ANU2,FUN)
      EXPX=ANU2
      FJX1=FUN*X**EXPX
      AK=X*(2.0-AI)
C
C        CONSTANT MULTIPLIER TIMES SECOND BESSEL FUNCTION
C
      FJX2(K)=(AK/(1.0-AI))*FJX1
C
C        CRITICAL LOAD FOR BEAM WITH WITH DESIGN BASED ON POWER OF N
```

```
C
      PCRIT(K)=(2.0-AI)*8.0*X
C
C        OPTIMUN LOAD  FOR BEAM WITH WITH DESIGN BASED ON POWER OF N
C
      POPT(K)=(AI+1.0)*PCRIT(K)
      IF (X.LT.6.0) GO TO 10
C
C        CALL DRAW AND USE INTERSECTION OF RESULTING CURVES AS SOLUTION
C
      CALL DRAW (601,AXIS,FJX2,1,0,LABEL1,ITITLE,0,0,0,0,0,0,6,6,1,LAST)
      CALL DRAW (601,AXIS,FJX,3,0,LABEL2,ITITLE,0,0,0,0,0,0,6,6,1,LAST)
  100 CONTINUE
      STOP
      END


      SUBROUTINE SUM(X,ANU,FUN)
C
C
C        THIS SUBROUTINE CALCULATES THE SUMMATION PORTION OF A NON-
C        INTEGER,  POSITIVE ORDER BESSEL FUNCTION.  THE ARGUMENT MULT-
C        IPLIER IS NOT INCLUDED SINCE IT IS USED IN THE MAIN PROGRAM.
C
C                             M 2M
C        INFINITY          (-1)  X
C          SUM       ----------------------------
C          M=0         (2M+N)
C                     2       M  GAMMA(NU+M+1)
C
C
      ARG=1.0+ANU
C
C        CALCULATE ERROR FUNCTION VALUE FOR M=0
C
      CALL GMMMA(ARG,DEN1,IER)
      EXP=ANU
      DEN2=2.0**EXP
      FUN=1.0/(DEN1*DEN2)
      APPROX=1.0E-2
      M=1
    5 IEXP1=2*M
      ANUM=(-1.0)**M*X**IEXP1
      EXP1=IEXP1
      EXP2=EXP1+ANU
      DEN1=2.0**EXP2
C
C        CALCULATE M FACTORIAL
```

```
C
      IFACT=1
      DO 10 I=1,M
   10 IFACT=IFACT*I
      DEN2=IFACT
      ARG=M+ANU+1
C
C        CALCULATE ERROR FUNCTION FOR POSITIVE M
C
      CALL GMMMA(ARG,DEN3,IER)
      DEN=DEN1*DEN2*DEN3
      FUN1=FUN
      FUN=FUN+ANUM/DEN
      M=M+1
      DIFF=ABS(FUN1-FUN)
      IF (DIFF.GT.APPROX) GO TO 5
      RETURN
      END


      SUBROUTINE GMMMA(XX,GX,IER)
      IF(XX-57.)6,6,4
    4 IER=2
      GX=1.E75
      RETURN
    6 X=XX
      ERR=1.0E-6
      IER=0
      GX=1.0
      IF(X-2.0)50,50,15
   10 IF(X-2.0)110,110,15
   15 X=X-1.0
      GX=GX*X
      GO TO 10
   50 IF(X-1.0)60,120,110
C
C        SEE IF X IS NEAR NEGATIVE INTEGER OR ZERO
C
   60 IF(X-ERR)62,62,80
   62 Y=FLOAT(INT(X))-X
      IF(ABS(Y)-ERR)130,130,64
   64 IF(1.0-Y-ERR)130,130,70
C
C        X NOT NEAR A NEGATIVE INTEGER OR ZERO
C
   70 IF(X-1.0)80,80,110
   80 GX=GX/X
      X=X+1.0
```

```
      GO TO 70
  110 Y=X-1.0
      GY=1.0+Y*(-0.5771017+Y*(+0.9858540+Y*(-0.8764218+Y*(+0.8328212+
     1Y*(-0.5684729+Y*(+0.2548205+Y*(-0.05149930)))))))
      GX=GX*GY
  120 RETURN
  130 IER=1
      RETURN
      END
```

2. SIMPLY SUPPORTED BEAM $(1-2x/\ell)^N$

```
      WRITE (6,150)
  150 FORMAT ('0',30X,'SIMPLY SUPPORTED BEAM SOLUTIONS',//)
      WRITE (6,225)
  225 FORMAT ('0',16X,'ANU',12X,'ANU2',14X,'K',12X,'PCRIT',13X,'POPT',//
     2)
      DO 100 I=990,997
      FJX=0.
      AI=I+1
      AI=0.001*AI
      ANU=(1.0-AI)/(4.0-2.0*AI)
      ANU2=(5.0-3.0*AI)/(4.0-2.0*AI)
      X=0.005
   10 X=X+0.0001
      CALL SUM (X,ANU,FUN)
      EXPX=ANU
      FJX=FUN*X**EXPX
      CALL SUM (X,ANU2,FUN)
      EXPX=ANU2
      FJX1=FUN*X**EXPX
      AK=X*(2.0-AI)
      FJX2=(AK/(1.0-AI))*FJX1
      DIFF=(FJX2-FJX)
      IF (DIFF.LT.0.0) GO TO 10
   50 CONTINUE
      PCRIT=(2.0-AI)*8.0*X
      POPT=(AI+1.0)*PCRIT
      WRITE (6,1000) ANU,ANU2,X,PCRIT,POPT
 1000 FORMAT (' ',10X,F10.4,6X,F10.4,6X,F10.5,6X,F10.5,6X,F10.5//)
  100 CONTINUE
      STOP
      END


      SUBROUTINE SUM(X,ANU,FUN)
      ARG=1.0+ANU
      CALL GMMMA(ARG,DEN1,IER)
      EXP=ANU
      DEN2=2.0**EXP
      FUN=1.0/(DEN1*DEN2)
      APPROX=1.0E-2
      M=1
    5 IEXP1=2*M
      ANUM=(-1.0)**M*X**IEXP1
      EXP1=IEXP1
      EXP2=EXP1+ANU
      DEN1=2.0**EXP2
      IFACT=1
      DO 10 I=1,M
```

```
   10 IFACT=IFACT*I
      DEN2=IFACT
      ARG=M+ANU+1
      CALL GMMMA(ARG,DEN3,IER)
      DEN=DEN1*DEN2*DEN3
      FUN1=FUN
      FUN=FUN+ANUM/DEN
      M=M+1
      DIFF=ABS(FUN1-FUN)
      IF (DIFF.GT.APPROX) GO TO 5
      RETURN
      END

      SUBROUTINE GMMMA(XX,GX,IER)
      IF(XX-57.)6,6,4
    4 IER=2
      GX=1.E75
      RETURN
    6 X=XX
      ERR=1.0E-6
      IER=0
      GX=1.0
      IF(X-2.0)50,50,15
   10 IF(X-2.0)110,110,15
   15 X=X-1.0
      GX=GX*X
      GO TO 10
   50 IF(X-1.0)60,120,110
C
C        SEE IF X IS NEAR NEGATIVE INTEGER OR ZERO
C
   60 IF(X-ERR)62,62,80
   62 Y=FLOAT(INT(X))-X
      IF(ABS(Y)-ERR)130,130,64
   64 IF(1.0-Y-ERR)130,130,70
C
C        X NOT NEAR A NEGATIVE INTEGER OR ZERO
C
   70 IF(X-1.0)80,80,110
   80 GX=GX/X
      X=X+1.0
      GO TO 70
  110 Y=X-1.0
      GY=1.0+Y*(-0.5771017+Y*(+0.9858540+Y*(-0.8764218+Y*(+0.8328212+
     1Y*(-0.5684729+Y*(+0.2548205+Y*(-0.05149930)))))))
      GX=GX*GY
  120 RETURN
```

```
130 IER=1                                                           GMMM 670
    RETURN                                                          GMMM 680
    END                                                             GMMM 690
```

3. SIMPLY SUPPORTED BEAM $(1+2x/\ell)^N$

```
      WRITE (6,150)
  150 FORMAT ('0',25X,'SIMPLY SUPPORTED BEAM - CASE (1+2X/L)'//)
      WRITE (6,225)
  225 FORMAT ('0',16X,'ANU',12X,'ANU2',17X,'X',12X,'PCRIT',13X,'POPT',//
     2)
      DO 100 I=1,10
      FJX=0.
C
C        CALCULATE THE VALUE OF THE POWER N WHERE N=AI
C
      AI=I-1
C
C        CHANGE THE THE POWER VALUE TO NON-INTEGER
C
      AI=0.1*AI
C
C        CALCULATE THE VALUE OF THE FIRST BESSEL FUNCTION ORDER
C
      ANU=(1.0-AI)/(4.0-2.0*AI)
C
C        CALCULATE THE VALUE OF THE SECOND BESSEL FUNCTION ORDER
C
      ANU2=(5.0-3.0*AI)/(4.0-2.0*AI)
C
C        USE VALUE OBTAINED FROM BESSEL EQ. CURVES FOR APPROX X
C
      X=0.99
      IF (I.LE.3) GO TO 10
      X=0.90
      IF (I.LE.5) GO TO 10
      X=0.78
      IF (I.LE.7) GO TO 10
      X=0.69
      IF (I.LE.8) GO TO 10
      X=0.58
      IF (I.LE.9) GO TO 10
      X=0.41
      IF (I.LE.10) GO TO 10
C
C        INCREMENT X FOR DESIRED ACCURACY
C
   10 X=X+0.0001
      CALL SUM (X,ANU,FUN)
      EXPX=ANU
C
C        FIRST BESSEL FUNCTION OF ARGUMENT K
C
      FJX=FUN*X**EXPX
```

```
      CALL SUM (X,ANU2,FUN)
      EXPX=ANU2
      FJX1=FUN*X**EXPX
      AK=X*(2.0-AI)
C
C        CONSTANT MULTIPLIER TIMES SECOND BESSEL FUNCTION
C
      FJX2=(AK/(1.0-AI))*FJX1
      DIFF=(FJX2-FJX)
      IF (DIFF.LT.0.0) GO TO 10
   50 CONTINUE
C
C        CRITICAL LOAD FOR BEAM WITH WITH DESIGN BASED ON POWER OF N
C
      PCRIT=(2.0-AI)*8.0*X
      AN=AI+1.0
      DNOM=2.0**AN-1.0
C
C        OPTIMUN LOAD  FOR BEAM WITH WITH DESIGN BASED ON POWER OF N
C
      POPT=AN/DNOM*PCRIT
      WRITE (6,1000) ANU,ANU2,X,PCRIT,POPT
 1000 FORMAT (' ',10X,F10.4,6X,F10.4,6X,F10.5,6X,F10.5,6X,F10.5//)
  100 CONTINUE
      STOP
      END


      SUBROUTINE SUM(X,ANU,FUN)
C
C
C        THIS SUBROUTINE CALCULATES THE SUMMATION PORTION OF A NON-
C        INTEGER,  POSITIVE ORDER BESSEL FUNCTION.  THE ARGUMENT MULT-
C        IPLIER IS NOT INCLUDED SINCE IT IS USED IN THE MAIN PROGRAM.
C
C                                 M 2M
C        INFINITY             (-1) X
C          SUM
C          M=0      -(2M+N)-----------------------
C                   2       M   GAMMA(NU+M+1)
C
C
      ARG=1.0+ANU
C
C        CALCULATE ERROR FUNCTION VALUE FOR M=0
C
      CALL GMMA(ARG,DEN1,IER)
      EXP=ANU
```

```
      DEN2=2.0**EXP
      FUN=1.0/(DEN1*DEN2)
      APPROX=1.0E-2
      M=1
    5 IEXP1=2*M
      ANUM=(-1.0)**M*X**IEXP1
      EXP1=IEXP1
      EXP2=EXP1+ANU
      DEN1=2.0**EXP2
C
C        CALCULATE M FACTORIAL
C
      IFACT=1
      DO 10 I=1,M
   10 IFACT=IFACT*I
      DEN2=IFACT
      ARG=M+ANU+1
C
C        CALCULATE ERROR FUNCTION FOR POSITIVE M
C
      CALL GMMMA(ARG,DEN3.IER)
      DEN=DEN1*DEN2*DEN3
      FUN1=FUN
      FUN=FUN+ANUM/DEN
      M=M+1
      DIFF=ABS(FUN1-FUN)
      IF (DIFF.GT.APPROX) GO TO 5
      RETURN
      END

      SUBROUTINE GMMMA(XX,GX,IER)
      IF(XX-57.)6,6,4
    4 IER=2
      GX=1.E75
      RETURN
    6 X=XX
      ERR=1.0E-6
      IER=0
      GX=1.0
      IF(X-2.0)50,50,15
   10 IF(X-2.0)110,110,15
   15 X=X-1.0
      GX=GX*X
      GO TO 10
   50 IF(X-1.0)60.120.110
C
C        SEE IF X IS NEAR NEGATIVE INTEGER OR ZERO
```

```
C                                                                       GMMM 500
   60 IF(X-ERR)62,62,80                                                 GMMM 510
   62 Y=FLOAT(INT(X))-X                                                 GMMM 520
      IF(ABS(Y)-ERR)130,130,64                                          GMMM 530
   64 IF(1.0-Y-ERR)130,130,70                                           GMMM 540
C                                                                       GMMM 550
C        X NOT NEAR A NEGATIVE INTEGER OR ZERO                          GMMM 560
C                                                                       GMMM 570
   70 IF(X-1.0)80,80,110                                                GMMM 580
   80 GX=GX/X                                                           GMMM 590
      X=X+1.0                                                           GMMM 600
      GO TO 70                                                          GMMM 610
  110 Y=X-1.0                                                           GMMM 620
      GY=1.0+Y*(-0.5771017+Y*(+0.9858540+Y*(-0.8764218+Y*(+0.8328212+   GMMM 630
     1Y*(-0.5684729+Y*(+0.2548205+Y*(-0.05149930)))))))                  GMMM 640
      GX=GX*GY                                                          GMMM 650
  120 RETURN                                                            GMMM 660
  130 IER=1                                                             GMMM 670
      RETURN                                                            GMMM 680
      END                                                               GMMM 690
```

4. CANTILEVER BEAM

```
      WRITE (6,150)
  150 FORMAT ('0',30X,'CANTILEVER BEAM SOLUTIONS',//)
      WRITE (6,250)
  250 FORMAT ('0',17X,'N',18X,'ANU',18X,'X',17X,'PCRIT',16X,'POPT',//)
      DO 100 I=1,10
      FJX=0.0
C
C        CALCULATE THE VALUE OF THE POWER N WHERE N=AI
C
      AI=I-1
C
C        CHANGE THE THE POWER VALUE TO NON-INTEGER
C
      AI=0.1*AI
C
C        CALCULATE THE VALUE OF THE FIRST BESSEL FUNCTION ORDER
C
      ANU=-(1.0-AI)/(4.0-2.0*AI)
      X=2.005
      IF (I.LE.4) GO TO 10
      X=2.100
      IF (I.LE.7) GO TO 10
      X=2.200
      IF (I.LE.10) GO TO 10
   10 X=X+0.0001
      CALL SUM (X,ANU,FUN)
      EXPX=ANU
C
C        FIRST BESSEL FUNCTION OF ARGUMENT K
C
      FJX=FUN*X**EXPX
      IF (FJX.GT.0.0) GO TO 10
C
C        CRITICAL LOAD FOR BEAM WITH WITH DESIGN BASED ON POWER OF N
C
      PCRIT=(2.0-AI)*X
C
C        OPTIMUN LOAD  FOR BEAM WITH WITH DESIGN BASED ON POWER OF N
C
      POPT=(AI+1.0)*PCRIT
      WRITE (6,1000) AI,ANU,X,PCRIT,POPT
 1000 FORMAT (' ',10X,F10.2,10X,F10.4,10X,F10.4,10X,F10.5,10X,F10.5//)
  100 CONTINUE
      STOP
      END
```

```
      SUBROUTINE SUM(X,ANU,FUN)
C
C
C        THIS SUBROUTINE CALCULATES THE SUMMATION PORTION OF A NON-
C        INTEGER, POSITIVE ORDER BESSEL FUNCTION.  THE ARGUMENT MULT-
C        IPLIER IS NOT INCLUDED SINCE IT IS USED IN THE MAIN PROGRAM.
C
C                                  M 2M
C        INFINITY            (-1)  X
C          SUM      ------------------------------------
C          M=0        (2M+N)
C                    2        M   GAMMA(NU+M+1)
C
C
      ARG=1.0+ANU
C
C        CALCULATE ERROR FUNCTION VALUE FOR M=0
C
      CALL GMMMA(ARG,DEN1,IER)
      EXP=ANU
      DEN2=2.0**EXP
      FUN=1.0/(DEN1*DEN2)
      APPROX=1.0E-2
      M=1
    5 IEXP1=2*M
      ANUM=(-1.0)**M*X**IEXP1
      EXP1=IEXP1
      EXP2=EXP1+ANU
      DEN1=2.0**EXP2
C
C        CALCULATE M FACTORIAL
C
      IFACT=1
      DO 10 I=1,M
   10 IFACT=IFACT*I
      DEN2=IFACT
      ARG=M+ANU+1
C
C        CALCULATE ERROR FUNCTION FOR POSITIVE M
C
      CALL GMMMA(ARG,DEN3,IER)
      DEN=DEN1*DEN2*DEN3
      FUN1=FUN
      FUN=FUN+ANUM/DEN
      M=M+1
      DIFF=ABS(FUN1-FUN)
      IF (DIFF.GT.APPROX) GO TO 5
      RETURN
```

```
      END

      SUBROUTINE GMMMA(XX,GX,IER)
      IF(XX-57.)6,6,4
    4 IER=2
      GX=1.E75
      RETURN
    6 X=XX
      ERR=1.0E-6
      IER=0
      GX=1.0
      IF(X-2.0)50,50,15
   10 IF(X-2.0)110,110,15
   15 X=X-1.0
      GX=GX*X
      GO TO 10
   50 IF(X-1.0)60,120,110
C
C        SEE IF X IS NEAR NEGATIVE INTEGER OR ZERO
C
   60 IF(X-ERR)62,62,80
   62 Y=FLOAT(INT(X))-X
      IF(ABS(Y)-ERR)130,130,64
   64 IF(1.0-Y-ERR)130,130,70
C
C        X NOT NEAR A NEGATIVE INTEGER OR ZERO
C
   70 IF(X-1.0)80,80,110
   80 GX=GX/X
      X=X+1.0
      GO TO 70
  110 Y=X-1.0
      GY=1.0+Y*(-0.5771017+Y*(+0.9858540+Y*(-0.8764218+Y*(+0.8328212+
     1Y*(-0.5684729+Y*(+0.2548205+Y*(-0.05149930)))))))
      GX=GX*GY
  120 RETURN
  130 IER=1
      RETURN
      END
```

5. GENERATE BEAM DESIGNS

```
      DIMENSION AXIS1(250),AXIS2(250),HP(250),HM(250),HPN(250),HMN(250)
      REAL LABEL/4H    /
      REAL*8 ITITLE(12)
      READ (5,2000) ITITLE
 2000 FORMAT (6A8)
      DO 100 N=1,10
      AN=N-1
      AN=0.1*AN
      AI=AN+1.0
      EL=100.0
      EL1=EL/2.0
      H=2.0
      DNOM=2.0**AI-1.0
      HO=(AI/DNOM)*H
      WRITE (6,5000) AN,HO
 5000 FORMAT (////' ',10X,F10.5,10X,F10.5////)
      X=0.0
      X1=0.0
      K=0
   10 K=K+1
      X=X+0.2
      X1=X1-0.2
   50 AXIS1(K)=X
      AXIS2(K)=X1
      HP(K)=0.5*HO*(1.0+(2.0*X)/EL)**AN
      HM(K)=-HP(K)
      HPN(K)=0.5*HO*(1.0-(2.0*X1)/EL)**AN
      HMN(K)=-HPN(K)
      IF (X.LT.49.8) GO TO 10
      X=X+0.199
      X1=X1-0.199
      K=K+1
      IF (X.LE.49.999) GO TO 50
      CALL DRAW (250,AXIS1,HP,1,0,LABEL,ITITLE,20.,2.0,2,4,2,2,8,4,1,LAS
     2T)
      CALL DRAW (250,AXIS1,HM,2,0,LABEL,ITITLE,20.,2.,2,4,2,2,8,4,1,LAST
     2)
      CALL DRAW (250,AXIS2,HPN,2,0,LABEL,ITITLE,20.,2.,2,4,2,2,8,4,1,LAS
     2T)
      CALL DRAW (250,AXIS2,HMN,3,0,LABEL,ITITLE,20.,2.,2,4,2,2,8,4,1,LAS
     2T)
  100 CONTINUE
      STOP
      END
```

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13. ABSTRACT

The treatise seeks to find the optimum shape of a thin beam under the influence of lateral buckling. The specific formulation is by means of the isoperimetric problem of the calculus of variations. An energy method approach yields the governing equations for the problem. An analytic solution was not obtained due to the nonlinearity and coupling conditions of the equations.

A variable height lateral buckling problem is formulated as an alternate attempt to find the optimum design. Through Bessel equation analysis numerous designs are obtained and the resulting buckling loads are calculated. The largest buckling load corresponds to a beam design which appears to be very close to the optimum shape.

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